Math 14
Winter 2009
Monday, January 12

A Sample ε-δ Proof

As an example of a formal ε-δ proof, we show that

$$\lim_{x \to 2} x^2 = 4.$$

Generally the proof will begin:
Let $\varepsilon > 0$ be given.
We must show there is a $\delta > 0$ such that

$$|x - 2| < \delta \implies |x^2 - 4| < \varepsilon.$$

At this point the proof can either proceed "backward," figuring out what δ will work, or "forward," saying, "Let $\delta = * * *$," and then proving that $|x - 2| < \delta \implies |x^2 - 4| < \varepsilon$. In either case, δ will probably be expressed in terms of ε.

Proof I:
Let $\varepsilon > 0$ be given.
We must show there is a $\delta > 0$ such that

$$|x - 2| < \delta \implies |x^2 - 4| < \varepsilon.$$

To see what δ we should pick, let's see how small $x^2 - 4$ is, in general, when x is within δ of 2. We look at two cases separately.

If $x \geq 2$ and $|x - 2| < \delta$ we have

$$2 \leq x < 2 + \delta$$

$$4 \leq x^2 < 4 + 4\delta + \delta^2$$
$$0 \leq x^2 - 4 < 4\delta + \delta^2$$

so in this case, if we choose δ so that

$$4\delta + \delta^2 \leq \varepsilon$$

we will have

$$0 < x^2 - 4 < 4\delta + \delta^2 \leq \varepsilon$$

$$|x^2 - 4| < \varepsilon.$$

If $x < 2$ and $|x - 2| < \delta$ we have

$$2 - \delta < x < 2$$

Provided $\delta < 2$ so that $2-\delta$ and x are both positive, we can square everything and get

$$4 - 4\delta + \delta^2 < x^2 < 4$$

$$\delta^2 - 4\delta < x^2 - 4 < 0$$

$$|x^2 - 4| < |\delta^2 - 4\delta| \leq |\delta^2| + |4\delta| = \delta^2 + 4\delta,$$

so again, if we choose δ so that

$$4\delta + \delta^2 \leq \varepsilon$$

we will have

$$|x^2 - 4| < 4\delta + \delta^2 \leq \varepsilon.$$

So we must have $\delta < 2$ and $4\delta + \delta^2 \leq \varepsilon$. We can make this second inequality hold as long as we make sure that both 4δ and δ^2 are less than $\frac{\varepsilon}{2}$. This will be guaranteed if we make sure that δ is less than both $\frac{\varepsilon}{8}$ and $\sqrt{\frac{\varepsilon}{2}}$.

Therefore we can choose any positive δ small enough so that

$$\delta < 2 \;\;\&\;\; \delta < \frac{\varepsilon}{8} \;\;\&\;\; \delta < \sqrt{\frac{\varepsilon}{2}}.$$

Proof II:
Let $\varepsilon > 0$ be given.
We must show there is a $\delta > 0$ such that

$$|x-2| < \delta \implies |x^2-4| < \varepsilon.$$

Let $\delta = \sqrt{\varepsilon+4} - 2$. [This is the δ from the back of the book. They arrived at it by applying the quadratic formula to the equation $4\delta + \delta^2 = \varepsilon$, which they found using reasoning like we used in Proof I.]

Show that

$$|x-2| < \delta \implies |x^2-4| < \varepsilon.$$

To see this, if $|x-2| < \delta$, we have

$$|x-2| < \delta \implies 2-\delta < x < 2+\delta \implies 4-\delta < x+2 < 4+\delta \implies -4-\delta < x+2 < 4+\delta$$

and therefore

$$|x+2| < |4+\delta| = 4+\delta$$

and we have

$$|x^2-4| = |(x+2)(x-2)| = |x+2||x-2| < \delta(4+\delta) = (\sqrt{\varepsilon+4}-2)(\sqrt{\varepsilon+4}+2) = \varepsilon,$$

which is what we need.