Math 14
Winter 2009
Homework Due Wednesday, February 4

For problems (1)-(3), define $\vec{F}(x,y) = \left(\frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2}\right)$.

(1.) Show that $\vec{F}$ satisfies the "mixed partials" test. That is, if $\vec{F} = (P,Q)$, then $\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}$.

(2.) Find the line integral of $\vec{F}$ around the counterclockwise oriented circle of radius r around the origin. (Note that the answer is not zero.)

(3.) Let $\vec{v} = (a,b)$. Find a vector $\vec{v}_\perp$ normal to $\vec{v}$, having the same length as (a,b), so that if you were standing on the xy-plane facing in the direction given by $\vec{v}$, the vector $\vec{v}_\perp$ would be pointing toward your right.

(4.) Here is a different formulation of line integral: If we consider a differential vector in the direction of the curve γ parametrized by the function $\vec{r}(t) = (x(t),\, y(t))$ to be

$$d\vec{r} = \vec{T}\,ds = \langle dx,\, dy\rangle = \langle x'(t),\, y'(t)\rangle\; dt,$$

where $\vec{T}$ is the unit tangent vector, then we can consider a differential vector in the direction normal to the curve γ (pointing directly across γ from left to right as you move along γ) to be

$$d\vec{r}_\perp = \vec{n}\,ds = \langle dy,\, -dx\rangle = \langle y'(t),\, -x'(t)\rangle\; dt,$$

where $\vec{n}$ is the unit normal vector pointing to the right. (Note, this is NOT the same unit normal vector $\vec{N}$ appearing in the expression for acceleration, because $\vec{N}$ always points toward the inside of the curve, whereas $\vec{n}$ always points toward the right, regardless of how γ curves.)

Then, just as

$$\int_\gamma \vec{F}\cdot\vec{T}\,ds = \int_a^b \vec{F}(x(t),\, y(t))\cdot\langle x'(t),\, y'(t)\rangle\; dt$$

is the path integral of the tangential component of $\vec{F}$ in the direction of a curve γ parametrized by the function $\vec{r}(t) = \langle x(t),\, y(t)\rangle$ for $a \le t \le b$,

$$\int_\gamma \vec{F}\cdot\vec{n}\,ds = \int_a^b \vec{F}(x(t),\, y(t))\cdot\langle y'(t),\, -x'(t)\rangle\; dt$$

is the path integral of the normal component of $\vec{F}$ in the left-to-right direction across γ.

Find the path integral of the normal component of $\vec{F}$ across the counter-clockwise oriented circle of radius r around the origin if:

(a.) $\vec{F}(x, y) = \langle x,\, y \rangle$.

(b.) $\vec{F}(x, y) = \langle -y,\, x \rangle$.

(c.) $\vec{F}(x, y) = \left\langle \dfrac{x}{x^2 + y^2},\, \dfrac{y}{x^2 + y^2} \right\rangle$.