Thirteen Ways of Looking at a Derivative

with apologies to Wallace Stevens
(Thirteen Ways of Looking at a Blackbird)
Math 14, Project 2

I
Among twenty snowy mountains,
The only moving thing
Was the eye of the blackbird.

Problem 1:

Consider the operator $D = d/dx$ as a function whose domain is all differentiable functions on the real line and whose range is functions. Show that D is linear. That is, $D(af + bg) = aDf + bDg$ where f and g are functions and a and b are scalars.

II
I was of three minds,
Like a tree
In which there are three blackbirds.

Problem 2:

Let S be the set of all functions of the form $f(x) = asin(x) + bcos(x)$. Show that D takes: S to S in a 1-1 and onto map.

it III
The blackbird whirled in the autumn winds.
It was a small part of the pantomime.

Problem 3:

Show that if we represent $f(x) = asinx + bcosx$ by the vector (a, b) then D can be represented on S as a two by two matrix, A_D.

IV
A man and a woman
Are one.
A man and a woman and a blackbird
Are one.

Problem 4:

Show that if we take P to be the set of polynomials in x of degree less than or equal to n, then D maps P to P and can be represented as a matrix, B_D.

V
I do not know which to prefer,
The beauty of inflections
Or the beauty of innuendoes,
The blackbird whistling
Or just after.

Problem 5:

Show that D is neither 1-1 nor onto on P.

VI
Icicles filled the long window
With barbaric glass.
The shadow of the blackbird
Crossed it, to and fro.
The mood
Traced in the shadow
An indecipherable cause.

Problem 6:

Define the matrix exponential function for a square n by n matrix N as

$$exp(tN) = I + tN + ... + \frac{t^k}{k!}N^k + ...$$

Show (by comparison with the usual exponential function) that the series converges for all N and t. That is, show that the series which represent each entry in the resulting matrix are convergent.

VII
O thin men of Haddam,
Why do you imagine golden birds?
Do you not see how the blackbird
Walks around the feet
Of the women about you?

Problem 7:
Show that $exp(tN)$ satisfies the differential equation

$$\frac{d}{dt}(exp(tN) = Nexp(tN)$$

Hint: differentiate the series term by term.

VIII
I know noble accents
And lucid, inescapable rhythms;
But I know, too,
That the blackbird is involved
In what I know.

Problem 8:
Let M be a square diagonal matrix. Compute $exp(tM)$ explicitly.

IX
When the blackbird flew out of sight,
It marked the edge
Of one of many circles.

Problem 9:
What is $exp(tA_D)$? Ah, the shock of recognition!

X
At the sight of blackbirds
Flying in a green light,
Even the bawds of euphony
Would cry out sharply.

Problem 10:
What is $exp(tB_D)$? Ah, again!

XI
He rode over Connecticut
In a glass coach.
Once, a fear pierced him,
In that he mistook
The shadow of his equipage
For blackbirds.

Problem 11:

Let A be an n by n matrix. Let $X(t)$ be a vector valued function and C be a vector of constants, both of size n by 1. Show that the vector valued function, $X(t) = exp(tA)C$, is a solution of $\frac{d}{dt}X(t) = AX(t)$ such that $X(0) = C$.

XII
The river is moving.
The blackbird must be flying.

Problem 12:

Let A be a two by two square matrix. You can think of A as a vector in R^4 Then $exp(tA)$ is a parametrized curve in four dimensional space. What is the tangent vector to $exp(tA)$ at $t = 0$?

XIII
It was evening all afternoon.
It was snowing
And it was going to snow.
The blackbird sat
In the cedar-limbs.

Problem 13:

Thinking of R^4 as the set of two by two matrices, define a vector field on this space by assigning to each two by two matrix M the vector AM. Here, A is a fixed two by two matrix. Show that $exp(tA)$ is a flow line of this vector field.