More Problems

Problem 1. Let A be an $m \times n$ matrix. Show that there is a constant M (depending on A) so that

$$||A\mathbf{x}|| \leq M||\mathbf{x}||$$

for all $\mathbf{x} \in \mathbb{R}^n$. *Hint:* Write $A\mathbf{x}$ as the sum involving the columns of A, according to the definition. Apply the triangle inequality to this sum and then use Cauchy-Schwarz.

Problem 2. Show that the function $f : \mathbb{R}^n \to \mathbb{R}$, defined by $f(\mathbf{x}) = ||\mathbf{x}||$, is continuous, either using the $\epsilon - \delta$ definition or by realizing f as the composition of two continuous functions.

For Problem 3 we will need the following fact, which is similar to Theorem 5 on p 121 of the text (compositions of continuous functions are continuous).

Lemma 1. *Let $f : A \subset \mathbb{R}^n \to \mathbb{R}^m$ and $g : B \subset \mathbb{R}^m \to \mathbb{R}^l$ be functions. Assume that $f(A) \subset B$. Furthermore, assume that $\lim_{\mathbf{x}\to\mathbf{x}_0} f(\mathbf{x}) = \mathbf{b}$ and that $\mathbf{b} \in B$. If g is continuous then*

$$\lim_{\mathbf{x}\to\mathbf{x}_0} g \circ f(\mathbf{x}) = g(\mathbf{b}).$$

Proof. Let $\epsilon > 0$. Since g is continuous at $\mathbf{b}$, there exists a $\delta_1 > 0$ so that if $||\mathbf{y} - \mathbf{b}|| < \delta_1$ and $\mathbf{y} \in B$ then $||g(y) - g(b)|| < \epsilon$. By the definition of the limit, we can also find a $\delta > 0$ so that if $0 < ||\mathbf{x} - \mathbf{x}_0|| < \delta$ and $\mathbf{x} \in A$ then $||f(\mathbf{x}) - \mathbf{b}|| < \delta_1$. Combining these two results, we find that for $0 < ||\mathbf{x} - \mathbf{x}_0|| < \delta$, $\mathbf{x} \in A$, we have $||f(\mathbf{x}) - \mathbf{b}|| < \delta_1$, $f(\mathbf{x}) \in B$, and so

$$||g(f(\mathbf{x})) - g(\mathbf{b})|| < \epsilon.$$

This finishes the proof. □

This can be briefly stated as follows. If

$$\lim_{\mathbf{x}\to\mathbf{x}_0} f(\mathbf{x})$$

exists and g is continuous then

$$\lim_{\mathbf{x}\to\mathbf{x}_0} g(f(x)) = g\left(\lim_{\mathbf{x}\to\mathbf{x}_0} f(\mathbf{x})\right).$$

That is, we can pull the limit sign "inside" the function.

Problem 3. Let $\mathbf{c} : [a, b] \to \mathbb{R}^n$ be a path, with coordinate functions $x_1, x_2, \ldots, x_n$ so that

$$\mathbf{c}(t) = (x_1(t), x_2(t), \ldots, x_n(t)).$$

According to the definition, if $\mathbf{c}$ is differentiable at a point t_0, then the derivatives $x_1'(t_0), x_2'(t_0), \ldots, x_n'(t_0)$ must all exist. Show that the converse holds. That is, show that if $x_1'(t_0), x_2'(t_0), \ldots, x_n'(t_0)$ all exist then $\mathbf{c}$ is differentiable at t_0. You will need to use Problem 2, the lemma above and part (v) of Theorem 3 on p 116.