# Some Examples of Limits

While the concept of limits may seem reasonable, the technical $\epsilon-\delta$ definition can be confusing until one has some experience with it. In order to gain some experience with this somewhat cumbersome definition we'll carefully work out some examples. We begin by restating the definition of a limit for convenience.

**Definition 1.** *Let $U \subset \mathbb{R}^n$ be an open set and let $f : U \to \mathbb{R}^m$ be a function with domain $U$. Let $\mathbf{x}_0$ be a vector in $U$ or on the boundary of $U$. Let $\mathbf{b} \in \mathbb{R}^m$. We say that the limit of $f$ as $\mathbf{x}$ approaches $\mathbf{x}_0$ is $\mathbf{b}$, written*

$$\lim_{\mathbf{x}\to\mathbf{x}_0} f(\mathbf{x}) = \mathbf{b},$$

*provided that for every $\epsilon > 0$ there is a $\delta > 0$ so that $||f(\mathbf{x}) - \mathbf{b}|| < \epsilon$ whenever $0 < ||\mathbf{x} - \mathbf{x}_0|| < \delta$ and $\mathbf{x} \in U$.*

**Remarks.**

- One should think of $\epsilon$ and $\delta$ as being small. In particular, the phrase "for every $\epsilon > 0$" should be thought of as saying "for every $\epsilon > 0$, *no matter how small*". With this in mind, the definition is a precise way of saying the following. As long as $\mathbf{x}$ is close enough to $\mathbf{x}_0$ (but not equal to it), it is possible to make $f(\mathbf{x})$ as close to $\mathbf{b}$ as we want.

- We mentioned this above, but when we take limits it is important that we never evaluate the function at the point $\mathbf{x}_0$ we are limiting to. It is the behavior of $f$ *around* $\mathbf{x}_0$ that we're interested in. For now we really don't care what actually happens *at* $\mathbf{x}_0$. This remark is especially important (or maybe obvious) when the limit point $\mathbf{x}_0$ is not in the domain of $f$, for then $f(\mathbf{x}_0)$ doesn't actually mean anything!

- When using the $\epsilon - \delta$ definition, $\delta$ almost always will depend on $\epsilon$. In fact, most proofs of the existence of limits using the $\epsilon - \delta$ definition run as follows. You start off by saying "Let $\epsilon > 0$". That is, we are picking *any* small number. We must produce a $\delta$ that "works" for this particular $\epsilon$. By this we mean that we must produce a $\delta$ so that if we have $0 < ||\mathbf{x} - \mathbf{x}_0|| < \delta$ then $||f(\mathbf{x}) - \mathbf{b}|| < \epsilon$. In general, we will actually get a formula for $\delta$ in terms of $\epsilon$. So, if someone came to you and said "Alright smartypants, how can you make $f(\mathbf{x})$ and $\mathbf{b}$ be within $10^{-50}$ of each other?" you could say "Well, according to my formula, any time we have $0 < ||\mathbf{x} - \mathbf{x}_0|| < 10^{-100}$ we'll have $||f(\mathbf{x}) - \mathbf{b}|| < 10^{-50}$. And don't call me smartypants."

- Finally, what's the point of having $U$ open and $\mathbf{x}_0$ being either in $U$ or on the boundary of $U$? The point is that we want to be able to let $\mathbf{x}$ get close to $\mathbf{x}_0$ and then see what happens to $f(\mathbf{x})$. But $f(\mathbf{x})$ only makes sense if $\mathbf{x}$ is in the domain of $f$. So we need to have $f$ defined on some set that has points that get arbitrarily close to $\mathbf{x}_0$. Taking $U$ to be open and $\mathbf{x}_0$ in $U$ or on the boundary of $U$ allows us to get close to $\mathbf{x}_0$ with points $\mathbf{x}$ where $f(\mathbf{x})$ is actually defined.

Now, on to the examples.

**Example.** Find

$$\lim_{(x,y)\to(0,0)} \frac{(x-y)^3}{x^2+y^2}.$$

**Solution.** Before we can even begin to attempt to verify the $\epsilon - \delta$ definition we need to know what the value of the limit is. We'll make an educated guess at the value and then check that we're right by rigorously using the definition.

For any $(x,y) \neq (0,0)$ we have

$$\frac{(x-y)^3}{x^2+y^2} = (x-y)\frac{x^2-2xy+y^2}{x^2+y^2}.$$

The fraction on the right looks ominously familiar to one that we encountered in class. It seems we should be able to use the same reasoning to show that it's bounded by 2. But the quantity $(x-y)$ certainly goes to zero as $(x,y)$ does, so we guess that the overall limit is probably 0.

Now we rigorously check it. First of all we have

$$\left|\frac{x^2-2xy+y^2}{x^2+y^2}\right| \leq \frac{x^2+y^2+2|xy|}{x^2+y^2} \leq \frac{2(x^2+y^2)}{x^2+y^2} = 2.$$

Here we have used the result $2|xy| \leq x^2+y^2$ which we deduced in class. It follows that

$$\left|\frac{(x-y)^3}{x^2+y^2}\right| = \left|(x-y)\frac{x^2-2xy+y^2}{x^2+y^2}\right| \leq 2|x-y| \leq 2(|x|+|y|).$$

Now for the $\epsilon - \delta$ stuff. Let $\epsilon > 0$ (I warned you we'd be saying that). Let $\delta = \epsilon/4$ (there's the formula for the needed $\delta$ in terms of the given $\epsilon$). Then if

$$0 < ||(x,y)|| < \delta$$

we have

$$2(|x|+|y|) \leq 2(\sqrt{x^2+y^2}+\sqrt{x^2+y^2}) = 4||(x,y)|| < 4\delta = \epsilon.$$

That is

$$\left|\frac{(x-y)^3}{(x^2+y^2)}\right| < \epsilon$$

whenever $0 < ||(x,y)|| < \delta$. So, for any given $\epsilon > 0$ we can find the $\delta$ needed to make the implication in the definition of limit true. We conclude that

$$\lim_{(x,y)\to(0,0)} \frac{(x-y)^3}{x^2+y^2} = 0.$$

□

**Example.** Find

$$\lim_{(x,y)\to(0,0)} \frac{x^2y}{\sqrt{2x^2+3y^2}}.$$

**Solution.** Again, we need some idea of what the limit is before we can actually prove anything. This time we use the inequalities

$$\left|\frac{x^2y}{\sqrt{2x^2+3y^2}}\right| \leq |x|\frac{|xy|}{\sqrt{2}\sqrt{x^2+y^2}} \leq \sqrt{x^2+y^2}\,\frac{(1/2)(x^2+y^2)}{\sqrt{2}\sqrt{x^2+y^2}} = \frac{(x^2+y^2)^{3/2}}{2^{3/2}}.$$

It should be clear now that as $(x,y) \to (0,0)$ our function tends to zero as well. So we try to prove that the limit is 0.

Let $\epsilon > 0$. Then set $\delta = 2^{1/2}\epsilon^{1/3}$. If

$$0 < ||(x,y)|| < \delta$$

then

$$\left|\frac{x^2y}{\sqrt{2x^2+3y^2}}\right| \leq \frac{(x^2+y^2)^{3/2}}{2^{3/2}} = \frac{||(x,y)||^3}{2^{3/2}} < \frac{\delta^3}{2^{3/2}} = \epsilon.$$

Hence, we have shown that

$$\lim_{(x,y)\to(0,0)} \frac{x^2y}{\sqrt{2x^2+3y^2}} = 0.$$

□

As we can see, the general strategy for dealing with limits when $(x,y) \to (0,0)$ is to *somehow* compare the function $f$ to $||(x,y)||$. This is often the hardest part of finding these kinds of limits, and once it's done the $\epsilon - \delta$ part is usually straightforward.

Now let's look at a few more complicated examples.

**Example.** Find

$$\lim_{(x,y)\to(0,0)} \frac{e^{xy}-1}{y}.$$

**Solution.** The key here is recognizing a familiar limit from single variable calculus hiding in this problem. We begin by writing

$$\frac{e^{xy}-1}{y} = x\frac{e^{xy}-1}{xy}$$

which is valid provided $x \neq 0$. As $(x, y) \to (0, 0)$ we certainly have $xy \to 0$. So we expect the fraction on the right to approach

$$\lim_{h\to 0} \frac{e^h - 1}{h}.$$

This is just the value of the derivative of $g(x) = e^x$ at $x = 0$. Since $g'(x) = e^x$, the limit above is just $e^0 = 1$. So we expect to have

$$\frac{e^{xy}-1}{y} = x\frac{e^{xy}-1}{xy} \to 0 \cdot 1 = 0$$

as $(x, y) \to (0, 0)$. We now check this using the $\epsilon - \delta$ definition of limits.

Let $\epsilon > 0$. We begin by choosing $\delta_1 > 0$ so small that

$$\left|\frac{e^h - 1}{h} - 1\right| < 1 \tag{1}$$

for $0 < |h| < \delta$, using the definition of the limit

$$\lim_{h\to 0} \frac{e^h - 1}{h} = 1.$$

We now set $\delta = \min\{\sqrt{\delta_1}, \epsilon/2\}$. Then for

$$0 < ||(x, y)|| < \delta$$

we have

$$|x| \leq \sqrt{x^2 + y^2} < \delta \leq \epsilon/2$$

and

$$|xy| \leq \frac{1}{2}(x^2 + y^2) < \delta^2 \leq \delta_1.$$

If we take $h = xy$ above, then equation (1) holds and so

$$\left|\frac{e^{xy}-1}{xy}\right| < 2.$$

Putting this all together gives

$$\left|\frac{e^{xy}-1}{y}\right| = |x| \left|\frac{e^{xy}-1}{xy}\right| < 2\frac{\epsilon}{2} = \epsilon$$

provided

$$0 < ||(x, y)|| < \delta.$$

Once again, we see that the definition of the limit is satisfied and we conclude that
$$\lim_{(x,y)\to(0,0)} \frac{e^{xy}-1}{y} = 0.$$
(Remark: technically we cheated just a little. Everything we did above is valid *provided* $x \neq 0$. What happens if $x = 0$? In this case
$$\frac{e^{xy}-1}{y} = 0.$$
Hence, we certainly have
$$\left|\frac{e^{xy}-1}{y}\right| < \epsilon$$
for points with $x = 0$ as well. ) □

At this point you may feel a little ripped off. After all, I said that we should be able to get a formula for $\delta$ in terms of $\epsilon$. But the $\delta$ given above depends on $\epsilon$ and on the quantity $\delta_1$. Haven't I cheated you? The answer is no...sort of. I invoked the definition of the limit to produce $\delta_1$, and if we really wanted $\delta_1$ in terms of $\epsilon$ we could appeal to a single variable calculus text and find a proof of the limit we used above. So, if we wanted to go through this extra step, we *could* get $\delta_1$, and hence $\delta$, in terms of $\epsilon$ alone.

Once we've established some results about continuity, some $\epsilon - \delta$ proofs can be simplified a bit. As in the example above, this is at the expense of not getting an explicit formula for $\delta$ in terms of $\epsilon$.

**Example.** Find
$$\lim_{(x,y)\to(0,0)} \frac{\cos(xy)-1}{x^2y^2}.$$

**Solution.** Since $xy \to 0$ as $(x, y) \to (0, 0)$, in order to attempt to figure out the value of this limit we need to first find
$$\lim_{h\to 0} \frac{\cos(h)-1}{h^2}.$$
This is not a derivative as before, but can be calculated using L'Hopital's rule since it is of the form $0/0$. We get
$$\lim_{h\to 0} \frac{\cos(h)-1}{h^2} = \lim_{h\to 0} -\frac{\sin(h)}{2h} = -\frac{1}{2}$$
from the (now familiar) fact that $\sin(\alpha)/\alpha \to 1$ as $\alpha \to 0$. So we guess that the value of the limit we are really interested in is $-1/2$. Now we check it using $\epsilon$'s and $\delta$'s.

Let $\epsilon > 0$. As before, we first choose $\delta_1 > 0$ so that

$$\left| \frac{\cos(h) - 1}{h^2} - \left(-\frac{1}{2}\right) \right| < \epsilon$$

whenever $0 < |h| < \delta_1$. Since $f(x, y) = xy$ is continuous everywhere

$$\lim_{(x,y)\to(0,0)} f(x, y) = f(0, 0) = 0.$$

Hence, we can find $\delta > 0$ so that for $0 < ||(x, y)|| < \delta$ we have

$$0 < |xy| < \delta_1.$$

(i.e. take $\epsilon = \delta_1$ in the $\epsilon - \delta$ definition of $\lim_{(x,y)\to(0,0)} f(x, y) = f(0, 0) = 0$). Then, any time we have $0 < ||(x, y)|| < \delta$ we also have $0 < |xy| < \delta_1$ so that

$$\left| \frac{\cos(xy) - 1}{(xy)^2} - \left(-\frac{1}{2}\right) \right| < \epsilon$$

by above. We conclude that

$$\lim_{(x,y)\to(0,0)} \frac{\cos(xy) - 1}{x^2y^2} = -\frac{1}{2}.$$

□

Okay, so now we've been able to show that a few limits exist and computed their values carefully using $\epsilon$'s and $\delta$'s. What about limits that fail to exist? For us, the typical way to detect this is to show that as we approach our limiting point from different directions we get different values. Let's look at an example of this.

**Example.** Show that

$$\lim_{(x,y)\to(0,0)} \frac{\cos x - 1 - (x^2/2)}{x^4 + y^4}$$

fails to exist.

**Solution.** Suppose we first approach $(0, 0)$ along the $y$-axis, that is, through points of the form $(y, 0)$, $y \neq 0$. Then

$$\frac{\cos x - 1 - (x^2/2)}{x^4 + y^4} = 0$$

so that if the limit did exist, it would have to be 0. But now suppose that we approach $(0, 0)$ along $y = x$, that is, through points of the form $(x, x)$, $x \neq 0$. Then

$$\frac{\cos x - 1 - (x^2/2)}{x^4 + y^4} = \frac{cos x - 1 - (x^2/2)}{2x^4}.$$

As we approach $(0,0)$ on this line we are thus forced to consider

$$\lim_{x \to 0} \frac{cosx - 1 - (x^2/2)}{2x^4}$$

which can be computed using L'Hopital's rule several times. Indeed,

$$\begin{aligned} \lim_{x \to 0} \frac{cosx - 1 - (x^2/2)}{2x^4} &= \lim_{x \to 0} \frac{-\sin x - x}{8x^3} \\ &= \lim_{x \to 0} \frac{-\cos x - 1}{24x^2} \\ &= \lim_{x \to 0} \frac{\sin x}{48x} \\ &= \frac{1}{48}. \end{aligned}$$

This means that if the limit exists, then it must equal $1/48$. Since $1/48 \neq 0$, we conclude that the limit as we approach $(0,0)$ does not exist in this case. □

To justify the procedure we just used we shall prove the following

**Theorem 1.** *Let $U \subset \mathbb{R}^n$ be open and let $f : U \to \mathbb{R}^m$ be a function. Let $\mathbf{x}_0$ be in $U$ or on the boundary of $U$ and let $\mathbf{b} \in \mathbb{R}^m$. Suppose that*

$$\lim_{\mathbf{x} \to \mathbf{x}_0} f(\mathbf{x}) = \mathbf{b}.$$

*Suppose further that we are given a continuous path $g : (a,b) \to \mathbb{R}^n$ so that $g(0) = \mathbf{x}_0$ and that*

$$\lim_{t \to 0} f(g(t)) = \mathbf{c}$$

*for some $\mathbf{c} \in \mathbb{R}^m$. Then $\mathbf{b} = \mathbf{c}$.*

The interpretation of this result is more important than its proof. The upshot is that *if* the function has a limit at a point and *if* when we approach that point on a curve the values of the function approach some limit as well, then these limits must be the same. Therefore, if we approach a point through two different curves and get two different values it must be the case that the limit in general at that point does not exist.

The proof is provided for completeness, but I don't expect you to read it unless you *really* want to. It's another pile of $\epsilon$'s and $\delta$'s.

*Proof.* We will show that for *every* $\epsilon > 0$ we have

$$||\mathbf{b} - \mathbf{c}|| < \epsilon.$$

This implies that $\mathbf{b} = \mathbf{c}$.

So, let $\epsilon > 0$. Using the definition of the limit, choose $\delta_1 > 0$ so that

$$||f(\mathbf{x}) - \mathbf{b}|| < \frac{\epsilon}{2}$$

whenever $0 < ||\mathbf{x} - \mathbf{x}_0|| < \delta_1$. Next, using continuity of $g$, choose $\delta_2 > 0$ so that

$$||g(t) - \mathbf{x}_0|| < \delta_1$$

whenever $0 < |t| < \delta_2$. Combining the prior inequalities gives

$$||f(g(t)) - \mathbf{b}|| < \frac{\epsilon}{2}$$

whenever $0 < |t| < \delta_2$. Finally, using the definition of the limit again, choose $\delta_3 > 0$ so that

$$||f(g(t)) - \mathbf{c}|| < \frac{\epsilon}{2}$$

whenever $0 < |t| < \delta_3$. Let $\delta = \min\{\delta_2, \delta_3\}$. Then for any $0 < |t| < \delta$ we have

$$||\mathbf{b} - \mathbf{c}|| = ||\mathbf{b} - f(g(t)) + f(g(t)) - \mathbf{c}|| \leq ||\mathbf{b} - f(g(t))|| + ||f(g(t)) - \mathbf{c}|| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

This is exactly what we wanted to show, so we're done. □