Lagrange Multipliers and The Implicit Function Theorem

The method of Lagrange multipliers is a consequence of the following theorem.

Theorem 1. *Let $U \subset \mathbb{R}^n$ be an open set and let $f : U \to \mathbb{R}$, $g : U \to \mathbb{R}$ be C^1 functions. Let $\mathbf{x}_0 \in U$, $c = g(\mathbf{x}_0)$, and let S be the level set of g with value c. Suppose that $\nabla g(\mathbf{x}_0) \neq 0$.*

If the function $f|_S$ has a local extremum at $\mathbf{x_0}$ then there is a $\lambda \in \mathbb{R}$ so that

$$\nabla f(\mathbf{x}_0) = \lambda g(\mathbf{x}_0).$$

The text provides a sketch of this proof that I find to be rather confusing. However, the primary ingredient in the proof is the Implicit Function Theorem, which the book doesn't prove but does state rather carefully. Here we will carefully prove the theorem on Lagrange multipliers by using the Implicit Function Theorem.

We begin with a definition.

Definition 1. *Let $U \subset \mathbb{R}^n$ be open and let $g : U \to \mathbb{R}$ be a C^1 function. Let $\mathbf{x}_0 \in U$, $c = g(\mathbf{x}_0)$ and let S be the level set of g with value c. If $\nabla g(\mathbf{x}_0) \neq \mathbf{0}$ then the* tangent space *to S at $\mathbf{x}_0$ is defined to be the set of all vectors $\mathbf{x} \in \mathbb{R}^n$ so that $\nabla g(\mathbf{x}_0) \cdot (\mathbf{x} - \mathbf{x}_0) = 0$.*

The tangent space to the level set S is "philosophically" all vectors orthogonal to $\nabla g(\mathbf{x}_0)$. However, if we want to have this space sit in the correct location in space (i.e. if we want it to actually be tangent to S) then we have to shift each of these vectors so that they emanate from the point $\mathbf{x}_0$. That is, if $\mathbf{v}$ is orthogonal to $\nabla g(\mathbf{x}_0)$ then we shift it to the correct location in space by adding $\mathbf{x}_0$. This results in the vector $\mathbf{v} + \mathbf{x}_0$, which can easily be shown to belong to the tangent space as defined above.

The next lemma will provide us with a means to deduce when two vectors are scalar multiples of one another. This result will be essential to our proof of the Lagrange multipliers theorem.

Lemma 1. *Let $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$ with $\mathbf{u} \neq \mathbf{0}$. Let T denote the set of all vectors $\mathbf{x} \in \mathbb{R}^n$ so that $\mathbf{x} \cdot \mathbf{u} = 0$. If $\mathbf{x} \cdot \mathbf{v} = 0$ for all $\mathbf{x} \in T$ then $\mathbf{v}$ is a scalar multiple of $\mathbf{u}$.*

Proof. Since $\mathbf{u} \neq \mathbf{0}$ we know that we can write

$$\mathbf{v} = \mathbf{v}_1 + \mathbf{v}_2$$

where $\mathbf{v}_1 = \alpha\mathbf{u}$ and $\mathbf{v}_2 \cdot \mathbf{u} = 0$ ($\mathbf{v}_1$ is just the orthogonal projection of $\mathbf{v}$ onto $\mathbf{u}$; that is, $\alpha = \mathbf{v} \cdot \mathbf{u}/||\mathbf{u}||^2$). By our assumption on $\mathbf{v}$ we must have

$$\begin{aligned} 0 &= \mathbf{v}_2 \cdot \mathbf{v} \\ &= \mathbf{v}_2 \cdot (\mathbf{v}_1 + \mathbf{v}_2) \\ &= \mathbf{v}_2 \cdot \mathbf{v_1} + ||\mathbf{v}_2||^2 \\ &= \alpha\mathbf{v}_2 \cdot \mathbf{u} + ||\mathbf{v}_2||^2 \\ &= 0 + ||\mathbf{v}_2||^2. \end{aligned}$$

It follows that $\mathbf{v}_2 = \mathbf{0}$ so that $\mathbf{v} = \mathbf{v}_1 = \alpha\mathbf{u}$ as claimed. □

Finally we can provide most of the proof of the Lagrange multiplier theorem.

Proof of Theorem 1. Let T be the tangent space to S at $\mathbf{x}_0$. We will show that for every $\mathbf{x} \in T$ we have $\nabla f(\mathbf{x}_0) \cdot (\mathbf{x} - \mathbf{x}_0) = 0$. Since $\nabla g(\mathbf{x}_0) \neq \mathbf{0}$, the lemma above will imply that $\nabla f(\mathbf{x}_0)$ is a scalar multiple of $\nabla g(\mathbf{x}_0)$, which is exactly what we need to prove.

We now state a result that we won't be able to prove until we have the Implicit Function Theorem at our disposal. Given any vector $\mathbf{v}$ with $\nabla g(\mathbf{x}_0) \cdot \mathbf{v} = 0$, there is a C^1 path $\mathbf{c} : [-a, a] \to \mathbb{R}^n$ so that

1. $\mathbf{c}(t) \in S$ for all $t \in [-a, a]$;
2. $\mathbf{c}(0) = \mathbf{x}_0$;
3. $\mathbf{c}'(0) = \mathbf{v}$.

What this is saying is that given a velocity vector $\mathbf{v}$ that is orthogonal to $\nabla g(\mathbf{x_0})$, we can find a C^1 path that stays entirely in the level set S and passes through $\mathbf{x}_0$ with velocity $\mathbf{v}$.

Now let $\mathbf{x} \in T$. Then $(\mathbf{x} - \mathbf{x}_0)$ is orthogonal to $\nabla g(\mathbf{x}_0)$ so we can choose a path $\mathbf{c}(t)$ as above. Let $h(t) = f(\mathbf{c}(t))$. Since $\mathbf{c}$ and f are both C^1, they are differentiable. By the chain rule h is differentiable, too. Since $\mathbf{c}(t) \in S$ for all t and $f|_S$ has a maximum or a minimum at $\mathbf{x}_0$, the function $h(t)$ has a maximum or a minimum at $t = 0$. Therefore

$$\begin{aligned} 0 &= \left.\frac{d}{dt} h(t)\right|_{t=0} \\ &= \nabla f(\mathbf{c}(0)) \cdot \mathbf{c}'(0) \\ &= \nabla f(\mathbf{x}_0) \cdot (\mathbf{x} - \mathbf{x}_0). \end{aligned}$$

This is exactly what we sought to prove, so we are finished.

So the proof of the Lagrange multipliers theorems is complete, provided we can justify the unproven fact that we used. This is where we need the Implicit Function Theorem, which we now state in a special case.

Theorem 2. *Let $F : \mathbb{R}^{n+1} \to \mathbb{R}$ be a C^1 function. Denoting points in $\mathbb{R}^{n+1}$ by $(\mathbf{x}, z)$, where $\mathbf{x} \in \mathbb{R}^n$ and $z \in \mathbb{R}$, assume that $(\mathbf{x}_0, z_0) \in \mathbb{R}^{n+1}$ satisfies*

$$F(\mathbf{x}_0, z_0) = 0 \text{ and } \frac{\partial F}{\partial z}(\mathbf{x}_0, z_0) \neq 0.$$

Then there is a ball $U \subset \mathbb{R}^n$ containing $\mathbf{x}_0$, an open interval $V \subset \mathbb{R}$ containing z_0 and a unique function $g : U \to \mathbb{R}$, so that $F(\mathbf{x}, z) = 0$ with $\mathbf{x} \in U$ and $z \in V$ if and only if $z = g(\mathbf{x})$. Moreover, the function g is C^1 with partial derivatives given by

$$\frac{\partial g}{\partial x_i} = -\frac{\partial F/\partial x_i}{\partial F/\partial z}$$

for $i = 1, 2, \cdots, n$.

This is a slight simplification of the statement given in the text. For a proof, take a look at the book's internet supplement.

We should think of the equation $F(\mathbf{x}, z) = 0$ as defining z implicitly as a function of $\mathbf{x}$. The theorem is just a precise formulation of this idea. It gives conditions, in terms of F, that tell us when we can write $z = g(\mathbf{x})$, and it tells us that g is C^1 if F is.

This result allows us to (finally!) finish the proof of the Lagrange multipliers theorem. Recall, the result we need to establish is the following.

Lemma 2. *Let $U \subset \mathbb{R}^n$ be open and let $g : U \to \mathbb{R}$ be a C^1 function. Let $\mathbf{x}_0 \in U$, $c = g(\mathbf{x}_0)$ and let S be the level set of g with value c. Assume that $\nabla g(\mathbf{x}_0) \neq \mathbf{0}$. Then given any vector $\mathbf{v}$ with $\nabla g(\mathbf{x}_0) \cdot \mathbf{v} = 0$, there is a C^1 path $\mathbf{c} : [-a, a] \to \mathbb{R}^n$ so that*

1. *$\mathbf{c}(t) \in S$ for all $t \in [-a, a]$;*
2. *$\mathbf{c}(0) = \mathbf{x}_0$;*
3. *$\mathbf{c}'(0) = \mathbf{v}$.*

Proof. Since $\nabla g(\mathbf{x}_0) \neq \mathbf{0}$, there is some x_i so that $\partial g/\partial x_i(\mathbf{x}_0) \neq 0$. For convenience let's assume that $\partial g/\partial x_n(\mathbf{x}_0) \neq 0$ (the other possibilities are handled similarly). To avoid double subscripts, we write $\mathbf{x}_0 = (y_1, y_2, \ldots, y_n)$. By the Implicit Function Theorem (applied with $F = g$ and $z = x_n$), there is a ball $U \subset \mathbb{R}^{n-1}$ containing $(y_1, y_2, \ldots, y_{n-1})$, an interval $V \subset \mathbb{R}$ containing y_n and a C^1 function $h(x_1, \ldots, x_{n-1})$ defined on U so that

$$g(x_1, x_2, \ldots, x_{n-1}, x_n) = c \text{ if and only if } x_n = h(x_1, x_2, \ldots, x_{n-1}). \tag{1}$$

That is, for points on the level set S near $\mathbf{x}_0$ we can solve for x_n in terms of $x_1, \ldots, x_{n-1}$. Hence, at least locally, S is the graph of some function.

Let $\mathbf{v} = (v_1, v_2, \ldots, v_n)$ be orthogonal to $\nabla g(\mathbf{x}_0)$. Now let

$$\mathbf{c}_1(t) = (y_1 + tv_1, y_2 + tv_2, \ldots, y_{n-1} + tv_{n-1})$$

and

$$\mathbf{c}(t) = (\mathbf{c}_1(t), h(\mathbf{c}_1(t))).$$

For small t, $\mathbf{c}_1(t)$ will lie in U and so we are guaranteed by equation (1) that

$$g(\mathbf{c}(t)) = g(\mathbf{c}_1(t), h(\mathbf{c}_1(t))) = c.$$

Therefore, the path $\mathbf{c}(t)$ lies entirely in the level set S, provided we keep t small. This gives part (1.) of the lemma.

As for part (2.), we have

$$\mathbf{c}(0) = (\mathbf{c}_1(0), h(\mathbf{c}_1(0))) = (y_1, y_1, \ldots, y_{n-1}, h(y_1, y_1, \ldots, y_{n-1})).$$

Again, equation (1) tells us that since $c = g(\mathbf{x}_0) = g(y_1, y_2, \ldots, y_n)$ we must have $y_n = h(y_1, y_1, \ldots, y_{n-1})$. So the equation above becomes $\mathbf{c}(0) = \mathbf{x}_0$, as needed.

Finally, we prove part (3.). By the chain rule

$$\begin{aligned}
\frac{d}{dt}\, h(\mathbf{c}_1(t))\big|_{t=0} &= \nabla h(\mathbf{c}_1(0)) \cdot \mathbf{c}_1'(0) && (2)\\
&= \nabla h(y_1, y_2, \ldots, y_{n-1}) \cdot (v_1, v_2, \ldots, v_{n-1}) && (3)\\
&= \frac{\partial h}{\partial x_1} v_1 + \frac{\partial h}{\partial x_2} v_2 + \cdots + \frac{\partial h}{\partial x_{n-1}} v_{n-1}. && (4)
\end{aligned}$$

The implicit function theorem tells us how to compute these partial derivatives of h. It gives

$$\frac{\partial h}{\partial x_1} v_1 + \frac{\partial h}{\partial x_2} v_2 + \cdots + \frac{\partial h}{\partial x_{n-1}} v_{n-1} = -\frac{1}{\partial g/\partial x_n}\left(\frac{\partial g}{\partial x_1} v_1 + \frac{\partial g}{\partial x_2} v_2 + \cdots + \frac{\partial g}{\partial x_{n-1}} v_{n-1}\right).$$

But the quantity in the parentheses is just $\nabla g(\mathbf{x}_0)\cdot\mathbf{v} - (\partial g/\partial x_n)v_n = -(\partial g/\partial x_n)v_n$, since $\mathbf{v}$ is orthogonal to $\nabla g(\mathbf{x}_0)$. The upshot of all of this is that

$$\frac{d}{dt}\, h(\mathbf{c}_1(t))\big|_{t=0} = -\frac{1}{\partial g/\partial x_n}\left(-\frac{\partial g}{\partial x_n}(\mathbf{x}_0) v_n\right) = v_n.$$

Therefore, when we differentiate $\mathbf{c}(t)$ at zero we find that

$$\mathbf{c}'(0) = (v_1, v_2, \ldots, v_{n-1}, v_n) = \mathbf{v}$$

which gives part (3.). □