Math 14

Supplementary Problem Set 2

1. Write down the solution to the system of equations which has the following row-reduced augmented matrix.

$$\left(\begin{array}{ccccccc|c} 1 & 0 & 0 & 0 & 2 & -1 & 0 & 4 \\ 0 & 0 & 1 & 0 & 2 & 0 & 0 & 2 \\ 0 & 0 & 0 & 1 & -1 & 1 & 0 & 3 \end{array}\right)$$

2. Is it true for a an n by n matrix M that if $MX = B$ has a unique solution for one column vector B, then $MX = C$ has a unique solution for every column vector C. What about for an m by n matrix where m need not equal n?

3. Is it true that each matrix has one and only one row reduced echelon form? Explain why or why not. (We talked about row reducing augmented matrices for systems of equations, but the idea applies to all matrices, whether augmented or not.) Note: We say A has the *row reduced form* B is B is a row reduced matrix that we can obtain by performing row operations of our three basic types on A.

4. An upper triangular matrix is defined in problem 19 on page 63 of our textbook. Prove or give a counter-example.
   (a) Every matrix may be row reduced to upper triangular form.
   (b) Every matrix may be row reduced to upper triangular form using only row operations of type $R_i + cR_j$ and cR_i.
   (c) Every matrix may be row reduced to upper triangular form using only row operations of type $R_i + cR_j$.

5. The transpose of a matrix A is the matrix whose rows are the columns of A and is denoted by A^t. For each of the following statements, prove it or give a counter-example.
   (a) $(A^t)^t = A$.
   (b) $AX = A^tX$ for every square matrix X.
   (c) $(A + B)^t = A^t + B^t$ whenever $A + B$ is defined.
   (d) $A^tB^t = (AB)^t$.
   (e) $(AB)^t = B^tA^t$.
   (f) If $AB = I$, then $A^tB^t = I$, when A is an n by n matrix and I is the n by n identity matrix.

6. If the system of equations has the solution $x_1 = a_1$, $x_2 = a_2$, $\dots$, $x_n = a_n$, then we can represent this solution as the column vector

$$X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}.$$

   Let X_p be a column vector that represents a solution to the system of equations represented by the matrix equation $AX = B$. Let X be a column vector representing a parameterization of the set of all solutions to the the system of equations represented by the matrix equation $AX = 0$. How can you find a parameterization of all solutions to the system of equations $AX = B$?

7. The rank of a matrix is the number of nonzero rows that a row reduced form of the matrix has. The nullity of a matrix A is the number of parameters needed to describe a general solution to the system of equations represented by $AX = 0$. What is the sum of the rank and the nullity of an m by n matrix and why?