Math 13, Winter 2017

Homework set 8, due Wed Mar 1

Please show your work. No credit is given for solutions without justification.

(1) Let $\mathcal{S}$ be the part of the surface $z = 2xy$ for pairs (x, y) with

$$x \geq 1\,,\ y \geq 1 \ \text{ and } \ x + y \leq 4 \Leftrightarrow y \leq 4 - x,$$

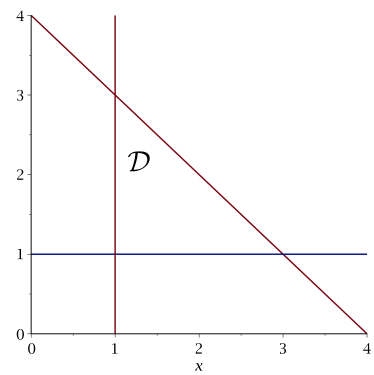
such that $\mathcal{S}$ is oriented by upward-pointing normal vectors.
For the vector field $\mathbf{F} = \langle -zx, zy, x\rangle$, calculate $\iint_{\mathcal{S}} \mathbf{F} \cdot dS$.

Solution:

Step 1: Finding a parametrization G for $\mathcal{S}$:
As $z = 2xy = h(x, y)$ is a function of x and y, the domain $\mathcal{D}$ of $h(x, y)$ will also be the domain of our parametrization G. From the above inequalities for x and y we get:

$$\mathcal{D} = \begin{cases} 1 \leq x \leq 3 \\ 1 \leq y \leq 4 - x \end{cases} \qquad \text{(see below)}.$$

Then $G(x, y) = (x, y, h(x, y)) = (x, y, 2xy)$.



Step 2: Calculating the normal vector $\mathbf{N}(x, y)$:
In general, the normal vector at a point (x, y, z) of a graph $z = h(x, y)$ is $\mathbf{N}(x, y) = (-\frac{\partial h(x,y)}{\partial x}, -\frac{\partial h(x,y)}{\partial y}, 1)$ (see p.941 of the book for this formula). In our case we get:

$$\mathbf{N}(x, y) = (-2y, -2x, 1).$$

Step 3: Substituting G in $\mathbf{F}$:
We get $\mathbf{F}(G(x, y)) = (-2x^2y, 2xy^2, x)$.

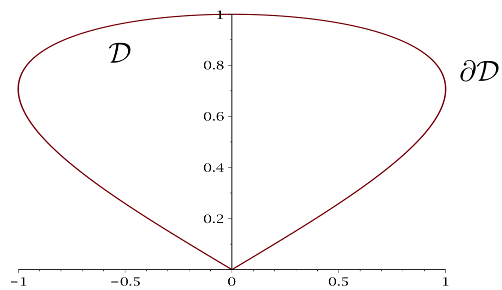
Step 4: Evaluating the integral $\iint_{\mathcal{S}} \mathbf{F} \cdot dS$:
As $\iint_{\mathcal{S}} \mathbf{F} \cdot dS = \iint_{\mathcal{D}} \mathbf{F}(G(x, y)) \bullet \mathbf{N}(x, y)\, dA$ we get:

$$\begin{aligned} \iint_{\mathcal{S}} \mathbf{F} \cdot dS &= \iint_{\mathcal{D}} 4x^2y^2 - 4x^2y^2 + x\, dA = \int_{x=1}^{3} \int_{y=1}^{4-x} x\, dy\, dx \\ &= \int_{x=1}^{3} x(4 - x - 1)\, dx = \int_{x=1}^{3} 3x - x^2\, dx \\ &= \frac{3}{2}x^2 - \frac{x^3}{3}\Big|_{x=1}^{3} = \frac{10}{3}. \end{aligned}$$

(2) Let $\mathcal{D}$ be the region depicted below, whose boundary $\partial\mathcal{D}$ is the so called *sinosoidal curve* with parametrization

$$\mathbf{r}(t) = (\sin(2t), \sin(t)), \quad 0 \leq t \leq \pi.$$

Find the area of $\mathcal{D}$.
Solution: We have seen three ways of calculating the area:



1.) $area(\mathcal{D}) = \int_{\partial\mathcal{D}} \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F} = \langle 0, x\rangle$.
2.) $area(\mathcal{D}) = \int_{\partial\mathcal{D}} \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F} = \langle -y, 0\rangle$.
3.) $area(\mathcal{D}) = \int_{\partial\mathcal{D}} \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F} = \langle -\frac{y}{2}, \frac{x}{2}\rangle$.

Using the first formula for the area, we get:

$$\mathbf{r}(t) = (\sin(2t), \sin(t)) = (x, y) \quad \text{and} \quad \mathbf{r}'(t) = (2\cos(2t), \cos(t)).$$

Note that by the trigonometric addition formulas

$$\sin(2t) = 2\sin(t)\cos(t) \quad \text{and} \quad \cos(2t) = \cos^2(t) - \sin^2(t) = 2\cos^2(t) - 1.$$

With $\mathbf{F} = \langle 0, x\rangle$, we have $\mathbf{F}(\mathbf{r}(t)) = \langle 0, \sin(2t)\rangle$. Hence

$$\begin{aligned}
area(\mathcal{D}) &= \int_{t=0}^{\pi} \mathbf{F}(\mathbf{r}(t)) \bullet \mathbf{r}'(t)\,dt \\
&= \int_{t=0}^{\pi} (0, \sin(2t)) \bullet (2\cos(2t), \cos(t))dt \\
&= \int_{t=0}^{\pi} \sin(2t) \cdot \cos(t)\,dt = \int_{t=0}^{\pi} (2\sin(t) \cdot \cos(t)) \cdot \cos(t)\,dt \\
&= \int_{t=0}^{\pi} 2\cos^2(t) \cdot \sin(t)\,dt = -\frac{2\cos^3(t)}{3}\Big|_{t=0}^{\pi} \\
&= \frac{2}{3} - (-\frac{2}{3}) = \frac{4}{3}.
\end{aligned}$$

If we apply the second formula for $area(\mathcal{D})$, we get:
With $\mathbf{F} = \langle -y, 0\rangle$, we have $\mathbf{F}(\mathbf{r}(t)) = \langle -\sin(t), 0\rangle$. Hence

$$\begin{aligned} area(\mathcal{D}) &= \int_{t=0}^{\pi} \mathbf{F}(\mathbf{r}(t)) \bullet \mathbf{r}'(t)\,dt \\ &= \int_{t=0}^{\pi} (-\sin(t), 0) \bullet (2\cos(2t), \cos(t))dt \\ &= \int_{t=0}^{\pi} -2\cos(2t)\cdot\sin(t)\,dt = \int_{t=0}^{\pi} -2\left(2\cos^2(t)-1\right)\cdot\sin(t)\,dt \\ &= \int_{t=0}^{\pi} 2\sin(t) - 4\cos^2(t)\sin(t)\,dt = -2\cos(t) + \frac{4\cos^3(t)}{3}\Big|_{t=0}^{\pi} \\ &= (2-\frac{4}{3}) - (-2+\frac{4}{3}) = \frac{4}{3}. \end{aligned}$$

Note that for $\mathbf{F} = \langle -\frac{y}{2}, \frac{x}{2}\rangle = \frac{1}{2}\langle 0, x\rangle + \frac{1}{2}\langle -y, 0\rangle$, $area(\mathcal{D}) = \int_{\partial\mathcal{D}} \mathbf{F}\cdot d\mathbf{r}$ can be calculated by combining the two previous cases.

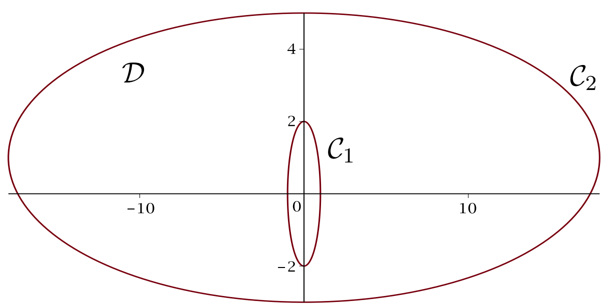
(3) Let $\mathbf{F}$ be the vector field in the xy-plane given by

$$\mathbf{F}(x,y) := \langle -\frac{3y}{4x^2+y^2}, \frac{3x}{4x^2+y^2}\rangle.$$

Let $\mathcal{D}$ be the region depicted below, whose boundary $\partial\mathcal{D}$ consists of the curves $\mathcal{C}_1$ and $\mathcal{C}_2$. Here $\mathcal{C}_1$ is the ellipse with parametrization

$$\mathbf{r}(t) = (\cos(t), 2\sin(t)),\quad 0 \le t \le 2\pi.$$

$\mathcal{C}_2$ is another closed curve, which is oriented, such that $\mathcal{D}$ is to the left of $\mathcal{C}_2$.



(a) Calculate $curl_z(\mathbf{F})$.
Solution:
Using the chain rule, we get:

$$curl_z(\mathbf{F}) = \frac{\partial\mathbf{F}_2}{\partial x} - \frac{\partial\mathbf{F}_1}{\partial y} = -\frac{3(4x^2-y^2)}{(4x^2+y^2)^2} - \left(-\frac{3(4x^2-y^2)}{(4x^2+y^2)^2}\right) = 0.$$

(b) Use the general form of Green's Theorem to find $\int_{\mathcal{C}_2} \mathbf{F}\cdot d\mathbf{r}$.
Solution:
The general form of Green's Theorem is $\int_{\partial\mathcal{D}} \mathbf{F}\cdot d\mathbf{r} = \iint_{\mathcal{D}} curl_z(\mathbf{F})\,dA$. As $curl_z(\mathbf{F}) = 0$ it follows that $\iint_{\mathcal{D}} curl_z(\mathbf{F})\,dA = 0$. With the given orientation of $\mathcal{C}_1$ and $\mathcal{C}_2$:

$$\int_{\partial\mathcal{D}} \mathbf{F}\cdot d\mathbf{r} = \int_{\mathcal{C}_2} \mathbf{F}\cdot d\mathbf{r} - \int_{\mathcal{C}_1} \mathbf{F}\cdot d\mathbf{r} = 0, \quad \text{hence} \quad \int_{\mathcal{C}_2} \mathbf{F}\cdot d\mathbf{r} = \int_{\mathcal{C}_1} \mathbf{F}\cdot d\mathbf{r}.$$

It is therefore sufficient to calculate $\int_{\mathcal{C}_1} \mathbf{F} \cdot d\mathbf{r}$. We get:

$$\mathbf{r}(t) = (\cos(t), 2\sin(t)) = (x, y) \quad \text{and} \quad \mathbf{r}'(t) = (-\sin(t), 2\cos(t)).$$

With $\mathbf{F} = \langle -\frac{3y}{4x^2+y^2}, \frac{3x}{4x^2+y^2} \rangle$, we have:

$$\mathbf{F}(\mathbf{r}(t)) = \langle -\frac{6\sin(t)}{4\cos^2(t) + 4\sin^2(t)}, -\frac{3\cos(t)}{4\cos^2(t) + 4\sin^2(t)} \rangle = \langle -\frac{6\sin(t)}{4}, -\frac{3\cos(t)}{4} \rangle.$$

Hence

$$\begin{aligned}
\int_{\mathcal{C}_1} \mathbf{F} \cdot d\mathbf{r} &= \int_{t=0}^{2\pi} \mathbf{F}(\mathbf{r}(t)) \bullet \mathbf{r}'(t)\, dt \\
&= \int_{t=0}^{2\pi} (-\frac{6\sin(t)}{4}, -\frac{3\cos(t)}{4}) \bullet (-\sin(t), 2\cos(t)) dt \\
&= \frac{3}{4} \int_{t=0}^{2\pi} 2(\sin^2(t) + \cos^2(t))\, dt = \frac{3}{2} \int_{t=0}^{2\pi} 1\, dt \\
&= 2\pi \cdot \frac{3}{2} = 3\pi.
\end{aligned}$$

In total we get: $\int_{\mathcal{C}_2} \mathbf{F} \cdot d\mathbf{r} = \int_{\mathcal{C}_1} \mathbf{F} \cdot d\mathbf{r} = 3\pi$.