Math 13, Winter 2017

Homework set 7, due Wed Feb 22

Please show your work. No credit is given for solutions without justification.

(1) Find an equation for the tangent plane of the cone $z = \sqrt{x^2 + y^2}$ at the point $(3, 4, 5)$.

Solution. First we need to find the normal vector to the cone. In general, the normal vector at a point (x, y, z) of a graph $z = f(x, y)$ is $\mathbf{N} = \langle -f_x, -f_y, 1\rangle$. (See p. 941 for this formula.) In our case, using the chain rule,

$$f_x = \frac{\partial f}{\partial x} = 2x \cdot \frac{1}{2}(x^2 + y^2)^{-1/2} = \frac{x}{\sqrt{x^2 + y^2}}$$

$$f_y = \frac{\partial f}{\partial y} = 2y \cdot \frac{1}{2}(x^2 + y^2)^{-1/2} = \frac{y}{\sqrt{x^2 + y^2}}$$

At the point $(3, 4, 5)$ we have $\sqrt{x^2 + y^2} = \sqrt{9 + 16} = 5$, so $f_x = \frac{3}{5}$ and $f_y = \frac{4}{5}$. The normal vector is $\mathbf{N} = \langle -\frac{3}{5}, -\frac{4}{5}, 1\rangle$.

We may rescale the normal vector: If $\langle -\frac{3}{5}, -\frac{4}{5}, 1\rangle$ is a normal vector, so is $\langle -3, -4, 5\rangle$. This rescaling is not necessary, but is a little easier to work with.

If the normal vector is $\mathbf{N} = \langle a, b, c\rangle$ then an equation for the tangent plane is $ax + by + cz = d$. So we find $-3x - 4y + 5z = d$ for the tangent plane. To find d we must make sure that the point $(3, 4, 5)$ is on the tangent plane. With $x = 3, y = 4, z = 5$ we get $d = -3x - 4y + 5z = -9 - 16 + 25 = 0$, so $d = 0$. An equation for the tangent plane is

$$-3x - 4y + 5z = 0$$

Remark. There are various equivalent forms of this equation. For example,

$$z = \frac{3}{5}x + \frac{4}{5}y$$

is also a correct equation of the tangent plane.

(2) What is the surface area of the part of the cylinder $x^2 + y^2 = 9$ that is above the xy-plane and below the plane $x + z = 3$?

Solution. Parametrize the cylinder in the standard way using cylindrical coordinates. With $r = 3$, $\theta = u$, $z = v$ we get $x = 3\cos u$, $y = 3\sin u$, $z = v$, or

$$G(u, v) = (3\cos u, 3\sin u, v)$$

The parameter domain $\mathcal{D}$ is determined by the equations $z \geq 0$ ("above the xy-plane") and $z \leq 3 - x$ ("below the plane $x + z = 3$"). Thus while $0 \leq u \leq 2\pi$ (since $u = \theta$) we have $0 \leq z \leq 3 - x$ or

$$0 \leq v \leq 3 - 3\cos u$$

Next we need to know the surface element $dS = \|\mathbf{N}\| du\, dv$. For a cylinder we have the standard formula $dS = R\, d\theta\, dz$ (see p. 940), which in our case gives $dS = 3du\, dv$.

We then find for the surface area,

$$\begin{aligned}\text{Area}(\mathcal{D}) &= \iint_{\mathcal{D}} 1\, dS = \int_0^{2\pi} \int_0^{3-3\cos u} 3\, dv\, du \\ &= \int_0^{2\pi} 9 - 9\cos u\, du = [9u - 9\sin u]_0^{2\pi} = 18\pi\end{aligned}$$

(3) Let $\mathcal{S}$ be the part of the sphere $x^2 + y^2 + z^2 = 4$ below the plane $z = 1$. Evaluate the surface integral $\iint_{\mathcal{S}} z^2 dS$.

Solution. 1. Parametrize the sphere using spherical coordinates. Since $\rho = 2$ we get

$$G(\theta, \phi) = (2\sin\phi\cos\theta, 2\sin\phi\sin\theta, 2\cos\phi))$$

If $z = 1$ then $2\cos\phi = 1$, or $\cos\phi = \frac{1}{2}$, which means that $\phi = \pi/3$. Therefore $z \leq 1$ corresponds to $\pi/3 \leq \phi \leq \pi$. The parameter domain $\mathcal{D}$ is determined by

$$0 \leq \theta \leq 2\pi, \quad \frac{\pi}{3} \leq \phi \leq \pi$$

2. We can use the standard formula for the area element dS of a sphere (on p. 940),

$$dS = \|\mathbf{N}\| d\phi\, d\theta = R^2 \sin\phi\, d\phi\, d\theta = 4\sin\phi\, d\phi\, d\theta$$

3. Finally, the integrand z^2 becomes $z^2 = 4\cos^2\phi$.

We find

$$\begin{aligned}\iint_{\mathcal{S}} z^2\, dS &= \int_0^{2\pi} \int_{\pi/3}^{\pi} 4\cos^2\phi \cdot 4\sin\phi\, d\phi\, d\theta = 16 \int_0^{2\pi} 1\, d\theta \cdot \int_{\pi/3}^{\pi} \cos^2\phi \sin\phi\, d\phi \\ &= 32\pi \cdot \int_{\pi/3}^{\pi} \cos^2\phi\sin\phi\, d\phi\end{aligned}$$

You do not need to consult wolframalpha to solve this integral. With the u-substitution $u = \cos\phi$ and $du = -\sin\phi\, d\phi$ we get

$$\int_{\pi/3}^{\pi} \cos^2\phi\sin\phi\, d\phi = \int_{\frac{1}{2}}^{-1} -u^2\, du = \left[-\frac{1}{3}u^3\right]_{u=\frac{1}{2}}^{-1} = \frac{1}{3} - (-\frac{1}{24}) = \frac{9}{24} = \frac{3}{8}$$

The final answer is

$$\iint_{\mathcal{S}} z^2\, dS = 32\pi \cdot \frac{3}{8} = 12\pi$$