Pset 6 (Solutions)

1. The vector field is

$$F = \frac{1}{x^2+y^2+z^2}\langle x, y, z\rangle$$

Let's check the cross partials

$$\frac{\partial F_1}{\partial y} = \frac{\partial}{\partial y}\left(\frac{x}{x^2+y^2+z^2}\right) = -x(x^2+y^2+z^2)^{-2}\cdot(2y)$$

$$= \frac{-2xy}{(x^2+y^2+z^2)^2}$$

$$\frac{\partial F_2}{\partial x} = \frac{\partial}{\partial x}\left(\frac{y}{x^2+y^2+z^2}\right) = \frac{-2xy}{(x^2+y^2+z^2)^2}$$

$$\Rightarrow \boxed{\frac{\partial F_1}{\partial y} = \frac{\partial F_2}{\partial x}}$$

Similarly, you will find that

$$\frac{\partial F_2}{\partial z} = \frac{\partial F_3}{\partial y} \quad \text{and} \quad \frac{\partial F_3}{\partial x} = \frac{\partial F_1}{\partial z}$$

Domain in this case is R^3 minus Origin $\Rightarrow$ Simply connected

$\Rightarrow$ F is conservative (Using Theorem 4 of Section 16.3)

To find potential function:

$$\frac{\partial f}{\partial x} = F_1 \Rightarrow f(x,y,z) = \int F_1\,dx$$

$$= \int \frac{x}{x^2+y^2+z^2}\,dx$$

$$= \frac{1}{2}\ln|x^2+y^2+z^2| + g(y,z) \quad -(1)$$

Similarly,

$$\frac{\partial f}{\partial y} = F_2 \Rightarrow f(x,y,z) = \int F_2\,dy$$

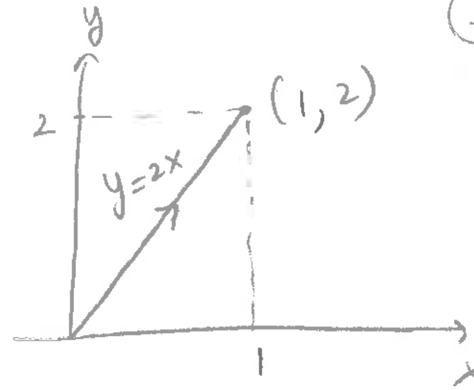
$$= \frac{1}{2}\ln|x^2+y^2+z^2| + h(x,z) \quad -(2)$$

$$\frac{\partial f}{\partial z} = F_3 \Rightarrow f(x,y,z) = \frac{1}{2}\ln|x^2+y^2+z^2| + k(x,y) - (3)$$

From (1), (2) and (3), we have.

$$\boxed{f(x,y,z) = \frac{1}{2}\ln|x^2+y^2+z^2|}$$

2 (a) $C_1 : y = 2x \qquad (0,0)$ to $(1,2)$



We use the parametrization

$$r(t) = (t,\ 2t), \quad 0 \le t \le 1$$

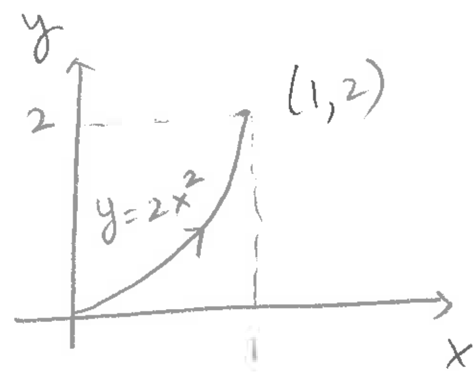
$$r'(t) = \langle 1, 2 \rangle$$

$$F = y^2 i + xy j = \langle y^2, xy \rangle \Rightarrow F(r(t)) = \langle 4t^2, 2t^2 \rangle$$

$$\int_{C_1} F \cdot dr = \int_0^1 \langle 4t^2, 2t^2 \rangle \cdot \langle 1, 2 \rangle \, dt$$

$$= \int_0^1 (4t^2 + 4t^2)\, dt = \int_0^1 8t^2\, dt = \frac{8t^3}{3}\Big|_0^1 = \frac{8}{3}$$

(b) $C_2 : y = 2x^2 \qquad (0,0)$ to $(1,2)$



We use the parametrization

$$r(t) = (t,\ 2t^2), \quad 0 \le t \le 1$$

$$r'(t) = \langle 1, 4t \rangle$$

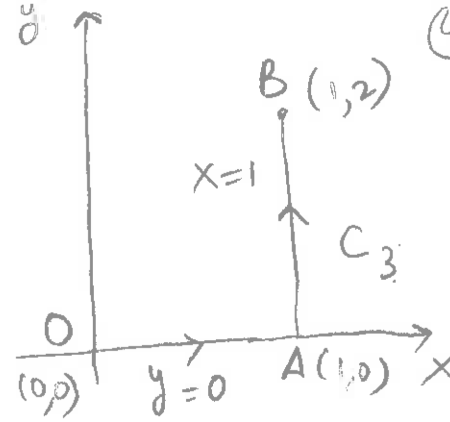
$$F = \langle y^2, xy \rangle \Rightarrow F(r(t)) = \langle 4t^4, 2t^3 \rangle$$

$$\int_{C_2} F \cdot dr = \int_0^1 \langle 4t^4, 2t^3 \rangle \cdot \langle 1, 4t \rangle \, dt$$

$$= \int_0^1 (4t^4 + 8t^4)\, dt = 12\int_0^1 t^4\, dt = 12\frac{t^5}{5}\Big|_0^1 = \frac{12}{5}$$

Note that the answer is different to part (a) $\Rightarrow$ line integral depends upon the path taken

(C) C_3: $y=0$ from $(0,0)$ to $(1,0)$
and $x=1$ from $(1,0)$ to $(1,2)$



Parametrization of $y=0$ from $(0,0)$ to $(1,0)$ is:

$r(t) = (t, 0) \qquad 0 \le t \le 1$

$r'(t) = \langle 1, 0 \rangle$

$F(r(t)) = \langle 0, 0 \rangle$

$$\int_{OA} F \cdot dr = \int_0^1 \langle 0,0 \rangle \cdot \langle 1,0 \rangle \, dt = 0 \qquad \text{--- } (1)$$

Parametrization of $x=1$ from $(1,0)$ to $(1,2)$

$r(t) = (1, t) \qquad 0 \le t \le 2$

$r'(t) = \langle 0, 1 \rangle$

$F(r(t)) = \langle t^2, t \rangle$

$$\int_{AB} F \cdot dr = \int_0^2 \langle t^2, t \rangle \cdot \langle 0, 1 \rangle \, dt$$

$$= \int_0^2 t \, dt = \left. \frac{t^2}{2} \right|_0^2 = 2 \qquad \text{--- } (2)$$

Using (1) and (2),

$$\int_C F \cdot dr = \int_{OA} F \cdot dr + \int_{AB} F \cdot dr = 0 + 2 = 2$$

Once again, the result is path dependent

3 The scalar line integral is

$$\int_C (3x + xy + z^3)\, ds$$

and the parametrization is

$$r(t) = (\cos 4t, \sin 4t, 3t), \quad 0 \le t \le 2\pi$$

$$r'(t) = (-4\sin 4t, 4\cos 4t, 3)$$

$$\|r'(t)\| = \sqrt{(-4\sin(4t))^2 + (4\cos 4t)^2 + (3)^2}$$

$$= \sqrt{16+9} = 5$$

$$f(r(t)) = 3(\cos 4t) + (\cos 4t)(\sin 4t) + (3t)^3$$

$$= 3\cos 4t + \cos 4t \sin 4t + 27t^3$$

So, the integral becomes

$$\int_C (3x + xy + z^3)\, ds = \int_0^{2\pi} (3\cos 4t + \cos 4t \sin 4t + 27t^3)\, 5\, dt$$

$$= 15 \int_0^{2\pi} \cos 4t\, dt + 5 \int_0^{2\pi} \cos 4t \sin 4t\, dt + 135 \int_0^{2\pi} t^3\, dt$$

$$= 15 \left. \frac{\sin(4t)}{4} \right|_0^{2\pi} + \frac{5}{2} \int_0^{2\pi} \sin 8t\, dt + 135 \left. \frac{t^4}{4} \right|_0^{2\pi}$$

$$= \frac{15}{4} \Big[\underbrace{\sin(8\pi)}_{0} - \underbrace{\sin(0)}_{0}\Big] + \frac{5}{2} \left. \left(\frac{-\cos 8t}{8}\right) \right|_0^{2\pi} + \frac{135}{4} (2\pi)^4$$

$$= \frac{5}{16}(-1+1) + 540\pi^4$$

$$= 540\pi^4$$