Math 13, Winter 2017

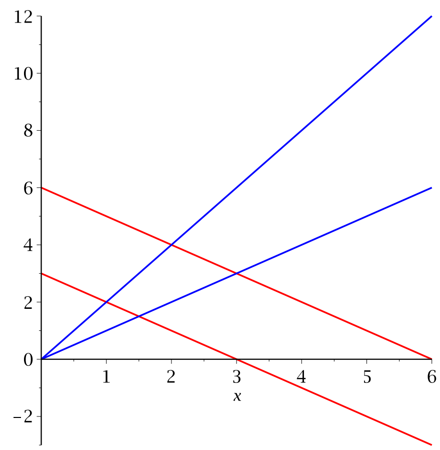
Homework set 5, due Wed Feb 8

Please show your work. No credit is given for solutions without justification.

(1) Let L_1 be the straight line passing through $(3,0)$ and $(0,3)$ and let L_2 be the line parallel to L_1 passing through $(6,0)$ and $(0,6)$. Let $\mathcal{D}$ be the region in the xy-plane enclosed by L_1 and L_2 and the lines $y = 2x$ and $y = x$. Let furthermore G be the map given by

$$G(u,v) := \left(\frac{u}{v+1}, \frac{u\cdot v}{v+1}\right) \quad \text{i.e.} \quad x = \frac{u}{v+1} \quad , \quad y = \frac{u\cdot v}{v+1}.$$

(a) Sketch $\mathcal{D}$ in the xy-plane.
Solution: We draw the different boundary lines and obtain $\mathcal{D}$ as the enclosed region



(b) Evaluate $\iint_{\mathcal{D}} x + y \; dA$ using a change of variables with the map G.
Solution:
Step 1: Finding the domain $\mathcal{D}_0$:
First we determine the region $\mathcal{D}_0$ in the uv-plane, such that $G(\mathcal{D}_0) = \mathcal{D}$.
We know: L_1 is given by the equation $y = 3 - x \Leftrightarrow x + y = 3$ and L_2 is given by the equation $y = 6 - x \Leftrightarrow x + y = 6$. Furthermore $y = 2x \Leftrightarrow \frac{y}{x} = 2$ and $y = x \Leftrightarrow \frac{y}{x} = 1$. Hence

$$\mathcal{D} = \begin{cases} 3 \le x + y \le 6 \\ 1 \le \frac{y}{x} \le 2 \end{cases}.$$

As $G(u,v) = (x,y)$, setting $x = \frac{u}{v+1}$, $y = \frac{u\cdot v}{v+1}$, we obtain in the above inequalities $x + y = u$ and $\frac{y}{x} = v$. The preimage of $\mathcal{D}$ in the uv-plane is therefore

$$\mathcal{D}_0 = \begin{cases} 3 \le u \le 6 \\ 1 \le v \le 2 \end{cases}, \quad \text{where} \quad G(\mathcal{D}_0) = \mathcal{D}.$$

Step 2: Compute the Jacobian:
For $G(u,v) = \left(\frac{u}{v+1}, \frac{u\cdot v}{v+1}\right) = (x(u,v), y(u,v))$ we have to calculate the Jacobian determinant of the matrix of derivatives:

$$Jac(G(u,v)) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \frac{1}{v+1} & \frac{-u}{(v+1)^2} \\ \frac{v}{(v+1)} & \frac{u}{(v+1)^2} \end{vmatrix} = \frac{u}{(v+1)^3} + \frac{uv}{(v+1)^3} = \frac{u}{(v+1)^2}.$$

Step 3: Substitute x and y in $f(x,y)=x+y$:
Since $x=\frac{u}{v+1}$ and $y=\frac{u\cdot v}{v+1}$ we get:

$$f(x,y)=x+y=\frac{u}{v+1}+\frac{u\cdot v}{v+1}=u.$$

Step 4: Apply the Change of Variables Formula:
Applying the Change of Variables Formula we get:

$$\begin{aligned}\iint_{\mathcal{D}} x+y\,dx\,dy=\iint_{\mathcal{D}_0} u\cdot\left|\frac{u}{(v+1)^2}\right|du\,dv &= \int_{v=1}^{2}\int_{u=3}^{6} u\cdot\frac{u}{(v+1)^2}\,du\,dv=\\ \int_{v=1}^{2}\frac{1}{(v+1)^2}\cdot\left(\int_{u=3}^{6}u^2\,du\right)dv &= 63\cdot\int_{v=1}^{2}\frac{1}{(v+1)^2}\,dv=\\ 63\cdot\frac{-1}{v+1}\bigg|_{v=1}^{2} &= \frac{21}{2}.\end{aligned}$$

(2) Let $F(x,y,z):=\left(\frac{x+z}{y},\frac{y}{z^2},\frac{z^2}{x^3}\right)$ be a vector field.

(a) Calculate $div(F)$.
Solution:
For $F(x,y,z):=(F_1,F_2,F_3)$ we have $div(F)=\nabla\bullet F=\frac{\partial F_1}{\partial x}+\frac{\partial F_2}{\partial y}+\frac{\partial F_3}{\partial z}$, where

$$\frac{\partial F_1}{\partial x}=\frac{\partial}{\partial x}\left(\frac{x+z}{y}\right)=\frac{1}{y}\quad,\quad\frac{\partial F_2}{\partial y}=\frac{\partial}{\partial y}\left(\frac{y}{z^2}\right)=\frac{1}{z^2}\quad,\quad\frac{\partial F_3}{\partial z}=\frac{\partial}{\partial z}\left(\frac{z^2}{x^3}\right)=\frac{2z}{x^3}.$$

Hence

$$div(F)=\frac{1}{y}+\frac{1}{z^2}+\frac{2z}{x^3}.$$

(b) Calculate $curl(F)$.
Solution:
For $F(x,y,z):=(F_1,F_2,F_3)$ we have

$$curl(F)=\nabla\times F=\left(\frac{\partial F_3}{\partial y}-\frac{\partial F_2}{\partial z},\frac{\partial F_1}{\partial z}-\frac{\partial F_3}{\partial x},\frac{\partial F_2}{\partial x}-\frac{\partial F_1}{\partial y}\right).$$

Hence

$$curl(F)=\left(0-\frac{-2y}{z^3},\frac{1}{y}-\frac{-3z^2}{x^4},0-\frac{(-x-z)}{y^2}\right)=\left(\frac{2y}{z^3},\frac{1}{y}+\frac{3z^2}{x^4},\frac{x+z}{y^2}\right).$$

(3) (Conservative vector fields and potential functions.)

(a) Let $F(x,y):=(x,y)$ and $G(x,y)=(-y,x)$ be two vector fields in the xy-plane. Find (by inspection) a potential function for F and prove that G is not conservative.
Solution:
For $f(x,y)=\frac{1}{2}\cdot(x^2+y^2)$ we have $\nabla f(x,y)=(x,y)=F(x,y)$.
For $G(x,y)=(G_1,G_2)$ we have:

$$\frac{\partial G_1}{\partial y}=\frac{\partial}{\partial y}(-y)=-1\neq 1=\frac{\partial}{\partial x}x=\frac{\partial G_2}{\partial x}.$$

Hence G can not be a conservative vector field by Ch. 16.1. **Theorem 1** of the book.

(b) Show that the vector field $H(x, y, z) := \big(2xyz, x^2z, x^2yz\big)$ in $\mathbb{R}^3$ does not have a potential function.

Solution:

We have that

$$\begin{aligned} curl(H) = \big(x^2z - x^2, 2xy - 2xyz, 2xz - 2xz\big) &= \\ (x^2(z-1), 2xy(1-z), 0) &\neq (0, 0, 0). \end{aligned}$$

Hence H can not be a conservative vector field by Ch. 16.1.**Theorem 1** of the book.