Math 13, Winter 2017

Homework set 4, due Wed Feb 1

Please show your work. No credit is given for solutions without justification.

(1) Suppose X and Y are two random variables with joint probability density function

$$p(x,y) = \begin{cases} 24xy & \text{if } x \geq 0, y \geq 0, \text{ and } x+y \leq 1; \\ 0 & \text{otherwise} \end{cases}$$

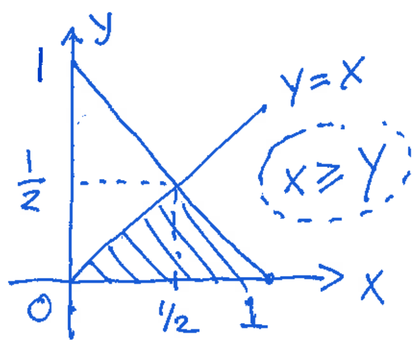
Calculate $P(X \geq Y)$, i.e. the probability that X is greater or equal to Y.

(2) Find the center of mass of the cylinder of radius 2 and height 4, i.e. the set of points with $x^2+y^2 \leq 4$ and $0 \leq z \leq 4$. The mass density is $\delta(x,y,z) = e^{-z}$.
Hint. You can use symmetry to find the x and y coordinates of the center of mass, without the need for calculations. Explain this.

(3) Apply the change of variables $x = 4u+2v$, $y = u+3v$ to the integral $\iint_{\mathcal{D}} x+y\,dA$, where $\mathcal{D}$ is the parallelogram with vertices $(0,0)$, $(8,2)$, $(12,8)$ and $(4,6)$, Then evaluate the integral.

Solutions

(1) The region where $p(x,y) \neq 0$ is the triangle with vertices $(0,0)$, $(1,0)$, $(0,1)$



The part of this triangle where $x \geq y$ is the region below the line $y=x$, which is the triangle with vertices $(0,0)$, $(1,0)$, $(\frac{1}{2},\frac{1}{2})$

This region is horizontally simple:

$$0 \leq y \leq 1/2$$

the line $y=x$ $\rightarrow$ $y \leq x \leq 1-y$ $\leftarrow$ the line $x+y=1$

Therefore

$$P(X \geq Y) = \int_0^{1/2} \int_Y^{1-Y} 24xy \, dx \, dy$$

To calculate the integral in two steps:

- $$\int_{x=Y}^{1-Y} 24xy\,dx = 12x^2y \Big|_{x=Y}^{1-Y} = 12\cancel{x}(1-y)^2y - 12y^3$$
$$= 12y - 24y^2 + 12y^3 - 12y^3$$
$$= 12y - 24y^2$$

- $$\int_0^{1/2} 12y - 24y^2 \, dy = 6y^2 - 8y^3 \Big|_0^{1/2} = \frac{6}{4} - \frac{8}{8} = \boxed{1/2}$$

The probability $P(X \geq Y) = 1/2$.

(2) The solid cylinder and the density function $\delta(x,y,z) = e^{-z}$ are symmetric for rotation around the z-axis. Therefore the center of mass must be on the z-axis, and we only need to calculate the z-coordinate.

Converting to cylindrical coordinates:

$$\iiint_W z\,\delta(x,y,z)\,dV = \int_0^{2\pi} \int_0^2 \int_0^4 ze^{-z} \cdot r \, dz \, dr \, d\theta$$

Evaluate in steps:

- $$\int_0^4 r z e^{-z}\,dz = \left[\begin{array}{c}\text{wolfram alpha} \\ \text{or integration by} \\ \text{parts}\end{array}\right]$$
$$= -r e^{-z}(z+1)\Big|_{z=0}^{4} = r\left(1-5e^{-4}\right)$$

- $$\int_0^2 r\left(1-5e^{-4}\right)dr = \left(1-5e^{-4}\right)\left[\tfrac{1}{2}r^2\right]_0^2$$
$$= 2-10e^{-4}.$$

- $$\int_0^{2\pi}\left(2-10e^{-4}\right)d\theta = 2\pi\left(2-10e^{-4}\right)$$

To find the total mass,

$$\iiint_W \delta(x,y,z)\,dV = \int_0^{2\pi}\int_0^2\int_0^4 e^{-z}\,r\,dz\,dr\,d\theta$$

In steps:

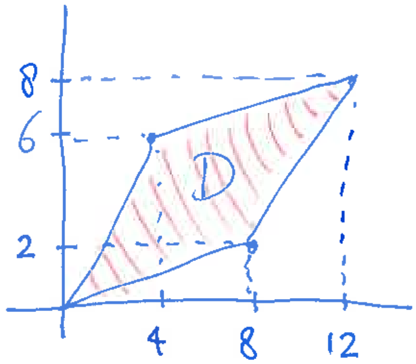
- $$\int_0^4 e^{-z} r\,dz = -re^{-z}\Big|_{z=0}^{4} = r\left(1-e^{-4}\right)$$

- $$\int_0^2 r\left(1-e^{-4}\right)dr = 2\left(1-e^{-4}\right)$$

- $$\int_0^{2\pi} 2-2e^{-4}\,d\theta = 2\pi\left(2-2e^{-4}\right)$$

$$\bar{z} = \frac{2\pi\left(2-10e^{-4}\right)}{2\pi\left(2-2e^{-4}\right)} = \boxed{\frac{1-5e^{-4}}{1-e^{-4}}} \approx 0.925\cdots$$

(3)



$$\begin{cases} x = 4u + 2v \\ y = u + 3v \end{cases}$$

First find the representatition of D in (u,v) plane. Must solve for (u,v) for each of the 4 vertices of the parallellogram.

Example: $x=4,\ y=6$ then
$$\begin{cases} 4 = 4u + 2v \\ 6 = u + 3v \end{cases}$$
times 4
$$24 = 4u + 12v$$
subtract
$$20 = 10v$$
$$\underline{v=2}$$

Substitute $v=2$ in $6 = u + 3v \Rightarrow \underline{u=0}$

So $(4,6) = (x,y)$ corresponds to $(u,v) = (0,2)$.

You can solve all 4 vertices this way.

Alternatively, derive a general formula:

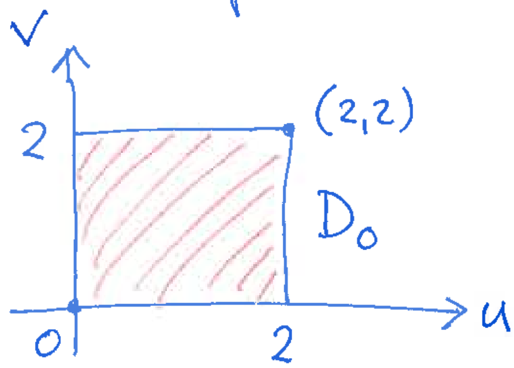
$$\begin{cases} x = 4u + 2v \\ y = u + 3v \end{cases} \xrightarrow{(\times 4)} \begin{matrix} x = 4u + 2v \\ 4y = 4u + 12v \end{matrix}$$
subtract
$$4y - x = 10v, \quad v = -\frac{1}{10}x + \frac{4}{10}y$$

Substitute in $y = u + 3v$ and get

$$\begin{cases} v = -\frac{1}{10}x + \frac{4}{10}y \\ u = \frac{3}{10}x - \frac{2}{10}y \end{cases}$$

Either way, we find the correspondence:

x	y	u	v
0	0	0	0
4	6	0	2
8	2	2	0
12	8	2	2



The substitution formulas are linear, so straight lines go to straight lines.

Now that we know D_0, use the Jacobian formula:

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix} = \begin{vmatrix} 4 & 2 \\ 1 & 3 \end{vmatrix} = 12 - 2 = 10$$

$$\Rightarrow \quad dx\,dy = 10\,du\,dv.$$

The function $f(x,y) = x+y$ converts to

$$x+y = (4u+2v) + (u+3v)$$
$$= 5u+5v.$$

All this assembles to the new integral

$$\underbrace{\int_0^2\int_0^2}_{D} \underbrace{5u+5v}_{f} \cdot \underbrace{10\,du\,dv}_{dA}$$

The integral works out as:

$$\int_0^2 \int_0^2 50u + 50v \, du\,dv$$

$$= \int_0^2 \left[25u^2 + 50uv\right]_{u=0}^{2} dv$$

$$= \int_0^2 (100 + 100v)\, dv$$

$$= 100v + 50v^2 \Big|_{v=0}^{2} = 200 + 200 = \boxed{400}$$