1(a) $$\int_{-1}^{2}\int_{0}^{\sqrt{4-x^2}} (x^2+y^2)\,dy\,dx$$

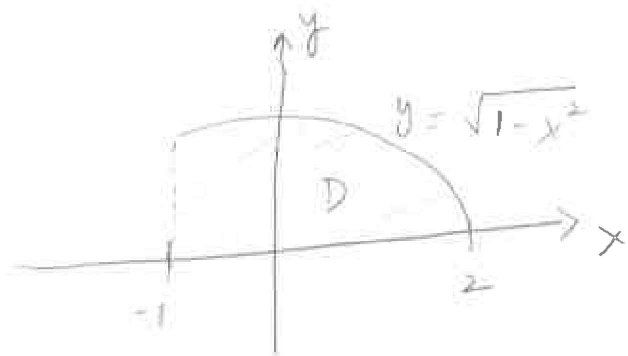


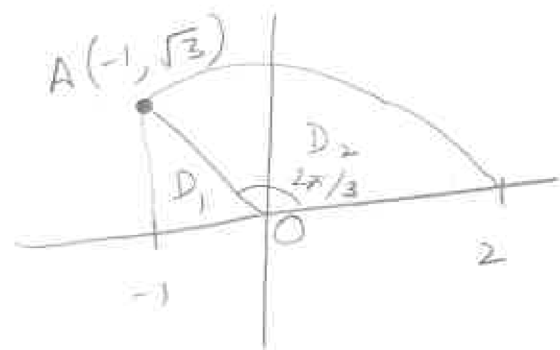
Fig. 1

The domain D is defined by the inequalities

$-1 \le x \le 2, \quad 0 \le y \le \sqrt{4-x^2}$

and is presented in Fig 1.

Now, we divide domain D into two domains D_1 and D_2 where D_1 is a triangle and D_2 is a circular section (see Fig. 2)



At $x=-1$, $y=\sqrt{3}$.

So, pt A is $(-1, \sqrt{3})$

Using properties of integrals, we have

$$\iint_D (x^2+y^2)\,dA = \iint_{D_1} (x^2+y^2)\,dA + \iint_{D_2} (x^2+y^2)\,dA \quad (1)$$

We compute each of these integrals separately and then add them up. Let's start with the integral over domain D_1

The line $x=-1$ has the polar co-ordinates $r\cos\theta = -1$

$\Rightarrow r = -\sec\theta$

Therefore, we have $0 \le r \le -\sec\theta$

The ray OA has polar eq.

$$r\cos\theta = -1$$

$$2\cos\theta = -1$$

$$\cos\theta = -1/2$$

$$\theta = 2\pi/3$$

So, we have $\frac{2\pi}{3} \le \theta \le \pi$

Thus, D_1 is described by

$$0 \le r \le -\sec\theta, \quad \frac{2\pi}{3} \le \theta \le \pi$$

By using change of variables, the integral over D_1 becomes

$$\iint_{D_1} (x^2+y^2)\, dA = \int_{\frac{2\pi}{3}}^{\pi} \int_0^{-\sec\theta} r^2 \cdot r\, dr\, d\theta = \int_{\frac{2\pi}{3}}^{\pi} \frac{r^4}{4}\Big|_0^{-\sec\theta} d\theta$$

$$= \int_{\frac{2\pi}{3}}^{\pi} \frac{(\sec\theta)^4}{4}\, d\theta$$

$$= \frac{1}{4}\int_{2\pi/3}^{\pi} \frac{1}{(\cos\theta)^4}\, d\theta$$

$$= \frac{1}{4}\int_{\frac{2\pi}{3}}^{\pi} \frac{\sin^2\theta + \cos^2\theta}{(\cos\theta)^4}\, d\theta$$

$$= \frac{1}{4}\int_{\frac{2\pi}{3}}^{\pi} \frac{\sin^2\theta}{\cos^4\theta}\, d\theta + \frac{1}{4}\int_{\frac{2\pi}{3}}^{\pi} \frac{1}{\cos^2\theta}\, d\theta$$

$$= \frac{1}{4}\int_{\frac{2\pi}{3}}^{\pi} \tan^2\theta \cdot \frac{1}{\cos^2\theta}\, d\theta + \frac{1}{4}\int_{\frac{2\pi}{3}}^{\pi} \sec^2\theta\, d\theta$$ (3)

$$= \frac{1}{4}\int_{\frac{2\pi}{3}}^{\pi} \tan^2\theta \sec^2\theta\, d\theta + \frac{1}{4}\tan\theta \Big|_{\frac{2\pi}{3}}^{\pi}$$

For the first integral, use u-substitution.

Let $u = \tan\theta$

$du = \sec^2\theta\, d\theta$

$\theta = \frac{2\pi}{3}$, $u = \tan\left(\frac{2\pi}{3}\right)$

$= -\sqrt{3}$

$\theta = \pi$, $u = \tan\pi = 0$

So,

$$\iint_{D_1} (x^2+y^2)\, dA = \frac{1}{4}\int_{-\sqrt{3}}^{0} u^2\, du + \frac{1}{4}\left[\tan(\pi) - \tan\left(\frac{2\pi}{3}\right)\right]$$

$$= \frac{1}{4}\frac{u^3}{3}\Big|_{-\sqrt{3}}^{0} + \frac{1}{4}\left[0 - (-\sqrt{3})\right]$$

$$= \frac{1}{4}\left[0 - \frac{(-\sqrt{3})^3}{3}\right] + \frac{\sqrt{3}}{4}$$

$$= \frac{1}{4}\frac{(\sqrt{3})^3}{3} + \frac{\sqrt{3}}{4} = \frac{1}{4}\frac{(\sqrt{3})^{\cancel{3}}}{\cancel{(\sqrt{3})^2}} + \frac{\sqrt{3}}{4}$$

$$= \frac{\sqrt{3}}{4} + \frac{\sqrt{3}}{4} = \frac{\sqrt{3}}{2}$$ — (2)

Domain D_2 is described by

$$0 \le \theta \le \frac{2\pi}{3}, \quad 0 \le r \le 2$$

Hence,

$$\iint_{D_2} (x^2+y^2)\, dA = \int_0^{2\pi/3} \int_0^2 r^2 \cdot r \, dr \, d\theta$$

$$= \int_0^{2\pi/3} \frac{r^4}{4}\Big|_0^2 \, d\theta$$

$$= \int_0^{2\pi/3} \frac{2^4}{4}\, d\theta = 4 \int_0^{2\pi/3} d\theta$$

$$= 4\theta \Big|_0^{2\pi/3}$$

$$= 4\left(\frac{2\pi}{3}\right)$$

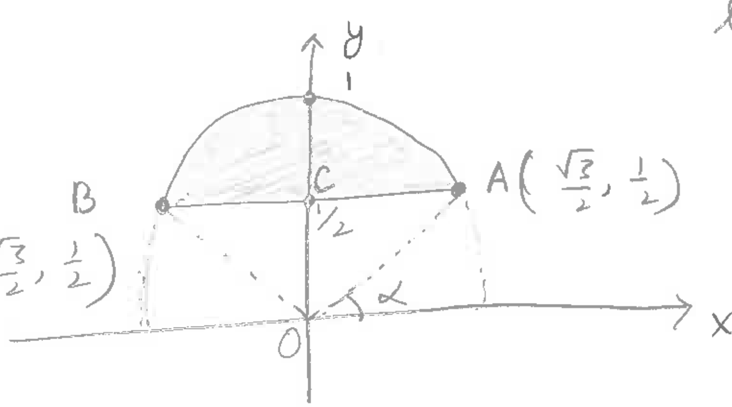
$$= \frac{8\pi}{3} \quad \text{—— } (3)$$

Combining equations (1), (2) and (3), we get

$$\boxed{\iint_D (x^2+y^2)\, dA = \frac{\sqrt{3}}{2} + \frac{8\pi}{3}}$$

(b) $f(x,y) = y\,(x^2+y^2)^{-1}$; $y \geq \frac{1}{2}$, $x^2+y^2 \leq 1$

The domain D is shown below. It is part of the unit circle lying above the line $y = \frac{1}{2}$



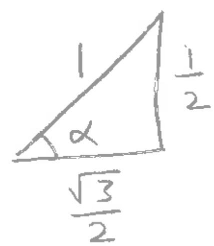
Using the fact that is a unit circle $\Rightarrow OA = 1$ and $OC = \frac{1}{2}$

$\Rightarrow AC = \frac{\sqrt{3}}{2}$

So, the pt A has the co-ordinates $A\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$

Similarly B has the co-ordinates $B\left(-\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$

So, the angle α is given by



$$\tan\alpha = \frac{\frac{1}{2}}{\frac{\sqrt{3}}{2}} = \frac{1}{\sqrt{3}}$$

$$\Rightarrow \alpha = \frac{\pi}{6}.$$

So, θ over the domain D varies from $\frac{\pi}{6}$ to $\pi - \frac{\pi}{6} = \frac{5\pi}{6}$

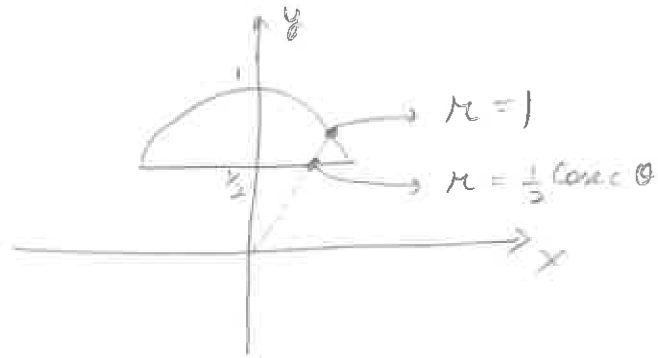
or, $\frac{\pi}{6} \leq \theta \leq \frac{5\pi}{6}$

The horizontal line $y = \frac{1}{2}$ has the polar equation

$$r \sin\theta = \frac{1}{2}$$

$$\Rightarrow \quad r = \frac{1}{2} \frac{1}{\sin\theta}$$

$$= \frac{1}{2} \operatorname{Cosec} \theta$$



The circle of radius 1, located at origin is given by

$$x^2 + y^2 = 1$$

In polar co-ordinates,

$$r^2 = 1 \Rightarrow r = 1$$

So,

$$\frac{1}{2} \operatorname{Cosec} \theta \le r \le 1$$

Therefore, the domain D is described by

$$\frac{1}{2} \operatorname{Cosec} \theta \le r \le 1, \quad \frac{\pi}{2} \le \theta \le \frac{5\pi}{6} \qquad \text{— (1)}$$

The function $f(x,y)$ in the integrand in polar co-ordinates,

$$f(x,y) = f(r\cos\theta, r\sin\theta) = (r\sin\theta)(r^2\cos^2\theta + r^2\sin^2\theta)^{-1}$$

$$= r\sin\theta \left(r^2(\cos^2\theta + \sin^2\theta)\right)^{-1}$$

$$= r\sin\theta (r^2)^{-1}$$

$$= \frac{\sin\theta}{r} \qquad \text{— (2)}$$

Using (1) and (2), we have

$$\iint_D y\,(x^2+y^2)^{-1}\,dA = \int_{\theta=\pi/6}^{5\pi/6} \int_{\frac{1}{2}\operatorname{Cosec}\theta}^{1} \frac{\sin\theta}{r} \cdot r\,dr\,d\theta$$

$$= \int_{\theta=\pi/6}^{5\pi/6} \int_{\frac{1}{2}\operatorname{Cosec}\theta}^{1} \sin\theta\,dr\,d\theta$$

$$= \int_{\theta=\pi/6}^{\theta=5\pi/6} r\sin\theta \Big|_{r=\frac{1}{2}\operatorname{Cosec}\theta}^{1} d\theta$$

$$= \int_{\theta=\pi/6}^{5\pi/6} \sin\theta \left[1-\frac{1}{2}\operatorname{Cosec}\theta\right] d\theta$$

$$= \int_{\theta=\pi/6}^{5\pi/6} \sin\theta\,d\theta - \frac{1}{2}\int_{\theta=\pi/6}^{5\pi/6} \sin\theta\operatorname{Cosec}\theta\,d\theta$$

$$= -\cos\theta \Big|_{\pi/6}^{5\pi/6} - \frac{1}{2}\int_{\pi/6}^{5\pi/6} d\theta$$

$$= \cos\left(\frac{\pi}{6}\right) - \cos\left(\frac{5\pi}{6}\right) - \frac{1}{2}\left[\frac{5\pi}{6}-\frac{\pi}{6}\right]$$

$$= \frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{2} - \frac{\pi}{3} = \sqrt{3} - \frac{\pi}{3}$$

2 (a) The region W is above the plane $z = 2$.

The plane $z = 2$ has cylindrical equation $z = 2$

W is also below the paraboloid $z = 6 - (x^2 - y^2)$

The paraboloid in cylindrical co-ordinates is $z = 6 - r^2$

The boundary of intersection of the plane and the paraboloid is

$$2 = 6 - (x^2 + y^2) \Rightarrow x^2 + y^2 = 4$$

so, the projection D of W onto the xy-plane is the circle of radius 2 centred at origin.

The polar equation of the circle is

$$D: \quad 0 \le \theta \le 2\pi, \quad 0 \le r \le 2$$

Therefore, the inequalities for W in cylindrical co-ordinates are

$$W: \quad 0 \le \theta \le 2\pi, \quad 0 \le r \le 2, \quad 2 \le z \le 6 - r^2$$

(b) Using the inequalities above, the volume of W is

$$\text{Vol}(W) = \iiint_W 1 \, dV = \int_{\theta=0}^{2\pi} \int_{r=0}^{2} \int_{z=2}^{6-r^2} 1 \cdot r \, dz \, dr \, d\theta$$

$$= \int_{\theta=0}^{2\pi} \int_{r=0}^{2} r z \Big|_{z=2}^{6-r^2} dr \, d\theta = \int_{\theta=0}^{2\pi} \int_{r=0}^{2} r\left[6 - r^2 - 2\right] dr \, d\theta$$

$$= \int_{\theta=0}^{2\pi} \int_{r=0}^{2} (4r - r^3)\, dr\, d\theta$$

$$= \int_{\theta=0}^{2\pi} \left(4\frac{r^2}{2} - \frac{r^4}{4} \right) \Bigg|_{r=0}^{2} d\theta$$

$$= \int_0^{2\pi} (8 - 4)\, d\theta = 4(2\pi) = 8\pi$$

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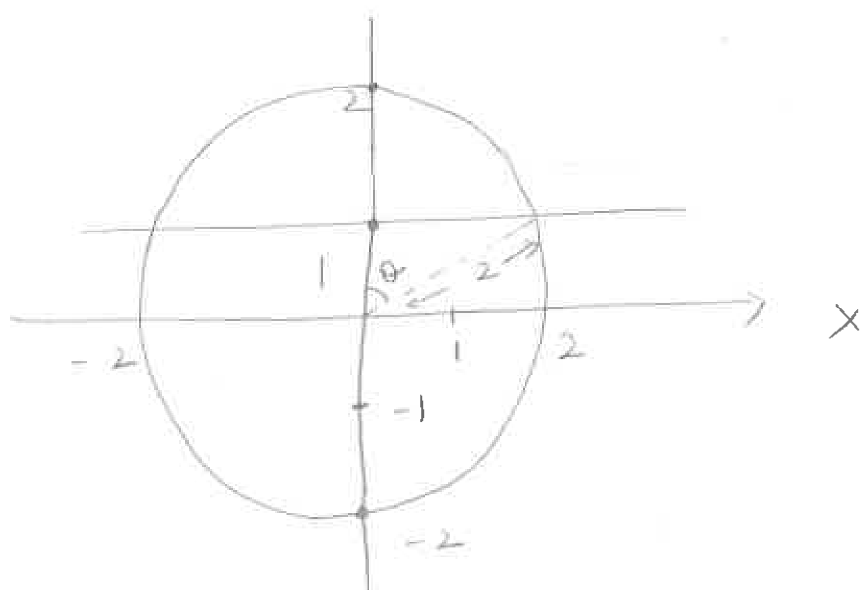
3 Since, the region is bounded below by the plane $z = 1$ and inside the sphere $x^2 + y^2 + z^2 = 4$, the diagram is below,



The angle φ, determined by the intersection of the sphere and the plane, satisfies

$$\cos\varphi = \frac{1}{2} \Rightarrow \boxed{\varphi = \frac{\pi}{3}}$$

The plane $z=1$ in spherical co-ordinates is

$$\rho \cos\varphi = 1$$

$$\Rightarrow \rho = \sec\varphi$$

The sphere $x^2+y^2+z^2=4$ in spherical coordinates is

$$\rho = 2$$

Thus, the region W is described by

$$W : \{ 0 \le \theta \le 2\pi,\ 0 \le \varphi \le \frac{\pi}{3},\ \sec\theta \le \rho \le 2 \}$$

The volume of the region is

$$\text{Vol.}(W) = \iiint_W 1\, dV = \int_0^{2\pi}\int_0^{\pi/3}\int_{\sec\varphi}^{2} \rho^2 \sin\varphi \, d\rho\, d\varphi\, d\theta$$

$$= \int_0^{2\pi}\int_0^{\pi/3} \left(\frac{1}{3}\rho^3 \sin\varphi\right)\Big|_{\sec\varphi}^{2} d\varphi\, d\theta$$

$$= \frac{1}{3}\int_0^{2\pi}\int_0^{\pi/3} \left(8\sin\varphi - \sec^3\varphi \sin\varphi\right) d\varphi\, d\theta$$

$$= \frac{1}{3}\int_0^{2\pi}\int_0^{\pi/3} \left(8\sin\varphi - \frac{\sin\varphi}{\cos^3\varphi}\right) d\varphi\, d\theta$$

$$= \frac{1}{3}\int_0^{2\pi} \left(-8\cos\varphi - \frac{1}{2\cos^2\varphi}\right)\Big|_0^{\pi/3} d\theta$$

$$= \frac{1}{3}\int_0^{2\pi} \left(-4 - \frac{1}{2(\frac{1}{4})} + 8 + \frac{1}{2}\right) d\theta$$

$$= \frac{1}{3}\int_0^{2\pi} \frac{5}{2}\, d\theta = \frac{5}{3}\pi$$