Math 13, Winter 2017

Homework set 2, due Wed Jan 18

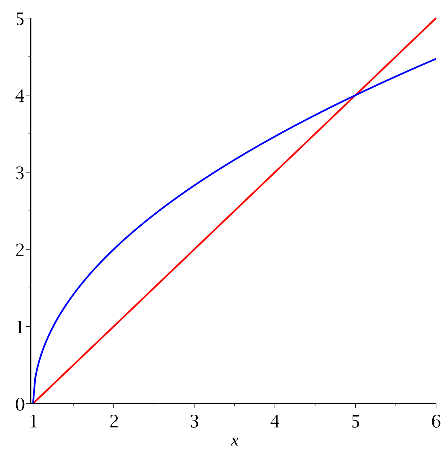
Please show your work. No credit is given for solutions without justification.

(1) Let $\mathcal{D}$ be the region in the xy-plane which lies above line $y = (x-1)$ and below the curve $y = 2\sqrt{x-1}$.

(a) Sketch $\mathcal{D}$ in the xy-plane and describe $\mathcal{D}$ as a horizontally simple region.
Solution: The two functions $f(x) = x - 1$ and $g(x) = 2\sqrt{x-1}$ intersect in $x \in \mathbb{R}$ where $g(x) = f(x)$. Hence

$$x - 1 = 2\sqrt{x-1} \Leftrightarrow (x-1)^2 = 4(x-1) \Leftrightarrow (x = 1) \quad \text{or} \quad (x = 5).$$

This can also be seen in the plot:



The coordinates of the intersection points are $(x, y) = (1, 0)$ and $(x, y) = (5, 4)$. To parametrize $\mathcal{D}$ as a horizontally simple region, we have to express the boundary curves in terms of y. We get:

$$y = (x-1) \Leftrightarrow x = (y+1) \quad \text{and} \quad y = 2\sqrt{x-1} \Rightarrow x = \frac{y^2}{4} + 1.$$

In total we get: $\mathcal{D} = \left\{ \begin{array}{l} 0 \leq y \leq 4 \\ \frac{y^2}{4} + 1 \leq x \leq y + 1 \end{array} \right.$.

(b) Evaluate $\iint_{\mathcal{D}}(x+\frac{y}{2})^2\,dA$ using the description from part (a).
Solution: It follows from part(a) that

$$\iint_{\mathcal{D}}(x+\frac{y}{2})^2\,dA=\int_{y=0}^{4}\left(\int_{x=\frac{y^2}{4}+1}^{y+1}(x+\frac{y}{2})^2\,dx\right)dy=$$

$$\int_{y=0}^{4}\frac{1}{3}\left((\frac{3y}{2}+1)^3-(\frac{y^2}{4}+\frac{y}{2}+1)^3\right)dy=$$

$$\frac{1}{3}\cdot\int_{y=0}^{4}(\frac{3y}{2}+1)^3-\left(\frac{1}{64}y^6+\frac{3}{32}y^5+\frac{3}{8}y^4+\frac{7}{8}y^3+\frac{3}{2}y^2+\frac{3}{2}y+1\right)dy=$$

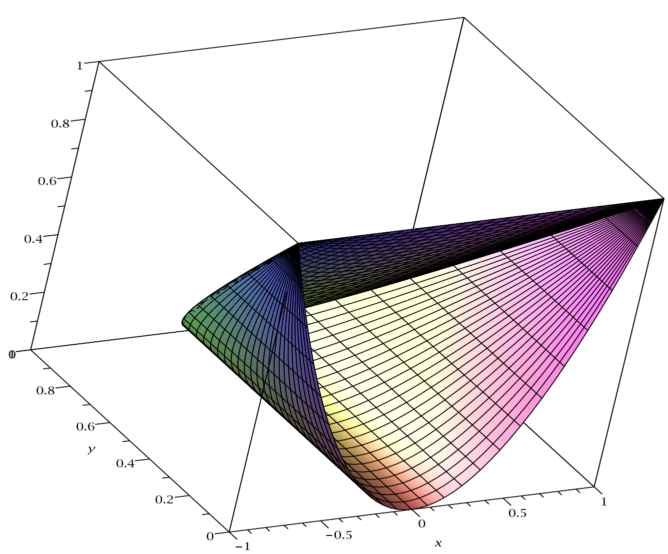
$$\frac{1}{3}\cdot\left(\frac{(\frac{3y}{2}+1)^4}{6}\Big|_0^4-\frac{9848}{35}\right)=\frac{1}{3}\cdot\left(400-\frac{9848}{35}\right)=\frac{1384}{35}.$$

(2) Let $\mathcal{W}$ be the solid region enclosed by the surfaces given by

$$z=x^2,\qquad z+y=1\ \text{ and }\ y=0.$$

Draw the region $\mathcal{W}$ then express $\iiint_{\mathcal{W}}1\,dV$ as an iterated integral in three different ways, by projecting onto each of the three different coordinate planes.
Note: You do NOT have to calculate the integral.

Solution:
Plot:



***xy*-plane:** We first gather some information. $\mathcal{W}$ is bounded by three intersecting surfaces. One is the xz-plane ($y=0$). The other two surfaces are given by the equations $z=x^2$ and $z+y=1$. If the two surfaces $z=x^2$ and $z+y=1\Leftrightarrow z=1-y$ intersect then

$$x^2=z=1-y\Leftrightarrow y=1-x^2.\qquad(*)$$

If we project onto the xy-plane, we have to describe the z-coordinates of $\mathcal{W}$ as a function of x and y. We have that $z = x^2$ and $z = 1 - y$. As we are looking for the enclosed region, we must have

$$x^2 \le z \le 1 - y.$$

We have already described the projection of the boundary line at the intersection of the two surfaces in Equation (*). This gives us the y-coordinates of one boundary of the domain $\mathcal{W}$ in terms of x:

$$y = 1 - x^2.$$

Note that we also could have expressed the x-coordinates of the boundary in terms of y. Finally we have to determine the interval for the x-coordinates. It is given by the intersection of $y = 1 - x^2$ with the plane $y = 0$. Hence

$$1 - x^2 = y = 0 \Leftrightarrow (x = -1) \quad \text{or} \quad (x = 1).$$

In total we get the following parametrization of $\mathcal{W}$:

$$-1 \le x \le 1, \quad 0 \le y \le 1 - x^2 \quad \text{and} \quad x^2 \le z \le 1 - y.$$

In a similar fashion we obtain for the other two projections:

yz-plane: $0 \le z \le 1, \quad 0 \le y \le 1 - z, \quad \text{and} \quad -\sqrt{z} \le x \le \sqrt{z}.$

xz-plane: $-1 \le x \le 1, \quad x^2 \le z \le 1, \quad \text{and} \quad 0 \le y \le 1 - z.$

(3) Let $\mathcal{W}$ be the tetrahedron in the first octant of space, whose vertices are

$$(0,0,0), \quad (0,0,4), \quad (2,0,0) \quad \text{and} \quad (0,2,0).$$

Let T be the temperature in this tetrahedron, given by $T(x,y,z) := 6y$ in degrees centigrade.

(a) Calculate the volume of $\mathcal{W}$.
Solution: First we have to parametrize the tetrahedron. The plane determined by the points $(0,0,4), \quad (2,0,0)$ and $(0,2,0)$ has the equation

$$z = 4 - 2x - 2y.$$

This result can be obtained by plugging in the points into the general equation of a plane $z = ax + by + c$ and then solving for a, b and c.
Then we have to parametrize the x and y coordinates of our domain $\mathcal{W}$. The intersection of $z = 4 - 2x - 2y$ with the xy-plane ($z = 0$) is

$$4 - 2x - 2y = 0 \Leftrightarrow y = 2 - x, \text{ hence } 0 \le y \le 2 - x.$$

Finally, the interval for the x-coordinates is $x \in [0, 2]$. In total we get:

$$\mathcal{W} = \begin{cases} 0 \le x \le 2 \\ 0 \le y \le 2 - x \\ 0 \le z \le 4 - 2x - 2y \end{cases}.$$

To find the volume we have to evaluate the integral

$$
\begin{aligned}
vol(\mathcal{W}) = \iiint_{\mathcal{W}} 1\ dV = \int_{x=0}^{2} \int_{y=0}^{2-x} \int_{z=0}^{4-2x-2y} 1\, dz\, dy\, dx &= \\
\int_{x=0}^{2} \int_{y=0}^{2-x} 4 - 2x - 2y\, dy\, dx &= \\
\int_{x=0}^{2} (4-2x)(2-x) - (2-x)^2\, dx &= \int_{x=0}^{2} (x-2)^2\, dx = \frac{8}{3}.
\end{aligned}
$$

(b) Calculate the average temperature in $\mathcal{W}$.
Solution: The average temperature $\bar{T}$ is $\bar{T} = \frac{1}{vol(\mathcal{W})} \cdot \iiint_{\mathcal{W}} T(x,y,z)\ dV$. We get:

$$
\begin{aligned}
\iiint_{\mathcal{W}} T(x,y,z)\ dV = \int_{x=0}^{2} \int_{y=0}^{2-x} \int_{z=0}^{4-2x-2y} 6y\, dz\, dy\, dx &= \\
\int_{x=0}^{2} \int_{y=0}^{2-x} 6y(4-2x-2y)\, dy\, dx &= \\
\int_{x=0}^{2} (12-6x)(2-x)^2 - 4(2-x)^3\, dx &= \int_{x=0}^{2} 2(2-x)^3\, dx = 8.
\end{aligned}
$$

In total we get for the average temperature in the tetrahedron:

$$
\bar{T} = \frac{3}{8} \cdot 8 = 3°C.
$$