1. $F = x^2y^3\,\hat{i} + \hat{j} + z\,\hat{k}$

In order to use Stokes' theorem, we calculate

$$\nabla \times F = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x^2y^3 & 1 & z \end{vmatrix} = -3y^2x^2\,\hat{k}$$

Using Stokes' theorem

$$\iint_S (\nabla \times F)\cdot ds = \iint_R -3y^2x^2\,dx\,dy \quad \text{where } R = \{(x,y) : x^2+y^2 \le 16\}$$

Using polar co-ordinates

$$= -3\int_0^{2\pi}\int_0^{4} r^4 \cos^2 \sin^2\theta \; r\,dr\,d\theta$$

$$= -3\int_0^{2\pi} (\cos\theta \sin\theta)^2 \left[\frac{r^6}{6}\right]\Bigg|_0^4 d\theta$$

$$= \frac{-4^6}{2\cdot 4}\int_0^{2\pi} (2\cos\theta \sin\theta)^2\,d\theta$$

$$= \frac{-4^5}{2}\int_0^{2\pi} (\sin 2\theta)^2\,d\theta \qquad \left[\because \sin 2\theta = 2\sin\theta\cos\theta\right]$$

Using u-substitution $u = 2\theta$, $du = 2d\theta$

and $\sin^2 u = \frac{1}{2}(1 - \cos 2u)$

$$= -\frac{4^5}{4}\int_0^{4\pi} \frac{(1-\cos 2u)}{2}\,du$$

$$= -4^4\left[\frac{u}{2} - \frac{\sin 2u}{4}\right]_0^{4\pi}$$

$$= -4^4\left[\frac{4\pi}{2}\right] = -\frac{4^5\pi}{2} = -512\pi$$

2. $F(x,y,z) = (x+y, z, x)$

$\text{div}(F) = \nabla \cdot F = 1 + 0 + 0 = 1$

and by Divergence theorem

$$\iint_S F \cdot ds = \iiint_W \text{div}(F)\, dv = \underbrace{\iiint_W 1\, dv}_{\text{volume of the hemisphere}}$$

$$= \frac{1}{2}\left[\frac{4}{3}\pi(1)^3\right] = \frac{2}{3}\pi$$

3 The vector field is

$$F = \langle y + e^{\sqrt{x}}, 2x + \cos(y^2) \rangle$$

$$\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} = 2 - 1 = 1$$

Thus by Green's Theorem

$$\int_C F \cdot ds = \int_0^1 \int_{x^2}^{\sqrt{x}} 1 \, dy \, dx = \int_0^1 y \Big|_{x^2}^{\sqrt{x}} dx$$

$$= \int_0^1 (\sqrt{x} - x^2) \, dx$$

$$= \left[\frac{2}{3} x^{3/2} - \frac{1}{3} x^3 \right]_0^1 = \frac{2}{3} - \frac{1}{3}$$

$$= \boxed{\frac{1}{3}}$$