Math 13 Winter 2017, Midterm Exam II
Tuesday February 14

Your name: ______________________________

Instructor (please circle): ☐ Bjoern Muetzel ☐ Vardayani Ratti ☐ Erik van Erp

INSTRUCTIONS

This is a closed book, closed notes exam.

You have 2 hours.

There are 5 multiple choice questions and 5 free response problems.

Use of calculators is not permitted.

The Honor Principle requires that you neither give nor receive any aid on this exam.

GOOD LUCK!

FERPA waiver: By my signature I relinquish my FERPA rights in the following context: This exam paper may be returned en masse with others in the class and I acknowledge that I understand my score may be visible to others.

FERPA waiver signature:

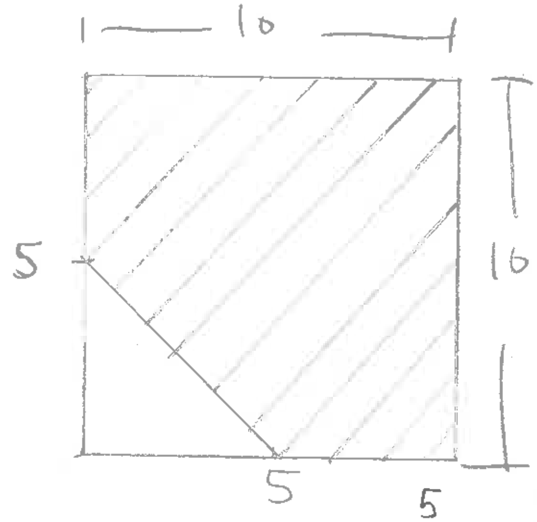
This page is for grading purposes only.

Problem	Points	Score
MC	30	
6	10	
7	10	
8	10	
9	10	
10	10	
Total	80	

Multiple Choice. *Each correct answer is worth 6 points. No points are subtracted for incorrect answers. You do not have to show your work for multiple choice questions. Extra paper is provided at the back of this booklet to do your calculations.*

(1) Real numbers X and Y between 0 and 10 are chosen randomly. The joint probability density is $p(x,y) = \frac{1}{100}$ if $0 \le x \le 10$ and $0 \le y \le 10$, and $p(x,y) = 0$ otherwise. What is the probability $P(X+Y > 5)$ that the sum of the two numbers is at least 5?

(A) $\frac{1}{2}$ (B) $\frac{3}{5}$ (C) $\frac{7}{10}$ (D) $\frac{3}{4}$ (E) $\frac{7}{8}$ (F) 1



• $P(0 \le X \le 10, 0 \le Y \le 10) = 1$

• $P(X+Y>5)$

$= 1 - P(X+Y \le 5)$

• $P(X+Y \le 5) = \int_{x=0}^{5}\int_{y=0}^{5-x} \frac{1}{100}\, dy\, dx$

$= \frac{1}{100}\int_{x=0}^{5} 5-x \, dx$

$= \frac{1}{100} \cdot 5x - \frac{x^2}{2}\Big|_0^5 = \frac{12.5}{100} = \frac{1}{8}$

$\Rightarrow P(X+Y>5) = 1 - \frac{1}{8} = \underline{\underline{\frac{7}{8}}}$

(2) Consider the curve $\mathcal{C}$ parametrized by $\mathbf{r}(t) = (t+1, t^4, 2t^2)$, where $0 \le t \le 1$, and the vector field $\mathbf{F} = \langle -4y, x, z \rangle$. Calculate $\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r}$.

(A) $\frac{1}{3}$ (B) $\frac{1}{2}$ (C) 1 (D) $\frac{3}{2}$ (E) 2 (F) 3

We have to calculate the vector line integral:

- $r(t) = (t+1, t^4, 2t^2) \quad t \in [0,1]$
- $r'(t) = (1, 4t^3, 4t)$
- $F(r(t)) = (-4t^4, t+1, 2t^2)$
- $F(r(t)) \bullet r'(t) = -4t^4 \cdot 1 + (t+1) \cdot 4t^3 + 2t^2 \cdot 4t$

$$= -4t^4 + 4t^4 + 4t^3 + 8t^3$$

$$= \underline{12t^3}$$

$$\int_C F \cdot dr = \int_{t=0} F(r(t)) \bullet r'(t)\, dt$$

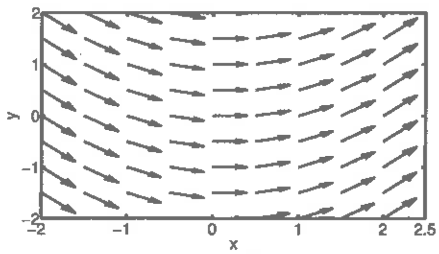
$$= \int_{t=0}^{1} 12t^3\, dt$$

$$= 12 \frac{t^4}{4} \Big|_{t=0}^{1} = \underline{\underline{3}}$$

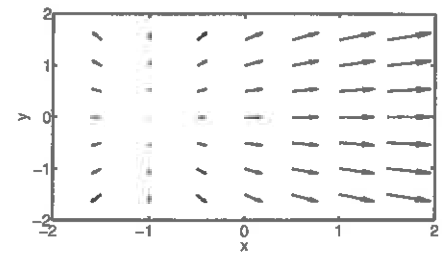
(3) Match each of the following vector fields with the corresponding plot.

(A) $\mathbf{F} = \langle 2, x \rangle$ (B) $\mathbf{F} = \langle 2x+2, y \rangle$ (C) $\mathbf{F} = \langle y, \cos x \rangle$ (D) $\mathbf{F} = \langle x+y, x-y \rangle$

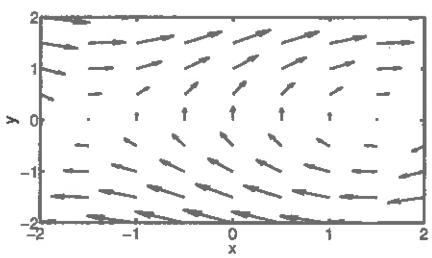
Mark one letter A, B, C, D in each of the empty boxes below the four plots.



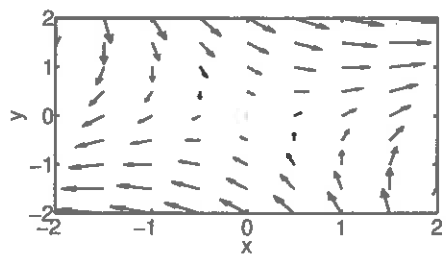
The top left plot: A



The top right plot: B



The bottom left plot: C



The bottom right plot: D

(4) Find the curl of $\mathbf{F} = \langle \frac{1}{2}y^2, \frac{1}{2}z^2, \frac{1}{2}x^2 \rangle$ at the point $(1,1,1)$.

(A) $\langle 1,1,1 \rangle$ (B) $\langle -1,1,1 \rangle$ (C) $\langle 1,-1,1 \rangle$ (D) $\langle 1,1,-1 \rangle$ (E) $\langle -1,1,-1 \rangle$ (F) $\langle -1,-1,-1 \rangle$

$$\text{curl } F = \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}, \frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}, \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right)$$

$$= (0 - z, 0 - x, 0 - y)$$

$$= (-z, -x, -y)$$

$$\Rightarrow \text{curl } F \text{ at } (1,1,1) \approx \underline{(-1,-1,-1)}$$

(5) Evaluate $\int_C z^2\, ds$ where C is the curve parametrized as

$$\mathbf{r}(t) = \left(\frac{1}{2}t^2, \frac{2\sqrt{2}}{3}t^{3/2}, t\right) \qquad 0 \le t \le 2$$

(A) 6 (B) $\frac{19}{3}$ (C) $\frac{20}{3}$ (D) 7 (E) $\frac{22}{3}$ (F) $\frac{23}{3}$

We know: $\bullet\ r'(t) = (t, \sqrt{2}\,t^{\frac{1}{2}}, 1) \quad 0 \le t \le 2$

$$\bullet\ \|r'(t)\| = (t^2 + 2t + 1)^{\frac{1}{2}} = ((t+1)^2)^{\frac{1}{2}} = \underline{t+1}$$

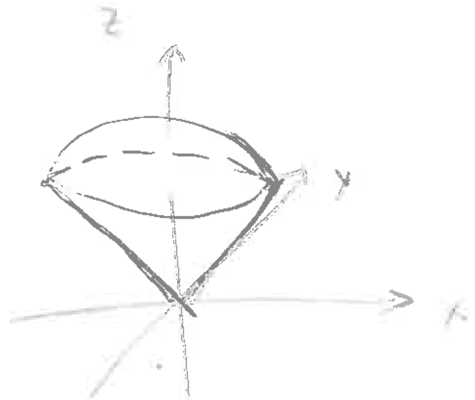
$$\bullet\ \int_C z^2\, ds = \int_{t=0}^{2} t^2 \cdot \|r'(t)\|\, dt = \int_{t=0}^{2} t^2 (t+1)\, dt = \int_{t=0}^{2} t^3 + t^2\, dt = \frac{t^4}{4} + \frac{t^3}{3} \Big|_{t=0}^{2} = 4 + \frac{8}{3} = \frac{12}{3} + \frac{8}{3} = \underline{\underline{\frac{20}{3}}}$$

Free Response. *Each free response question is worth 10 points. On each question you must show your work. No credit is given for solutions without supporting calculations. You will get partial credit for partially correct answers.*

(6) Let $\mathcal{W}$ be the ice-cream cone shaped solid region above the cone $z^2 = x^2 + y^2$ and inside the sphere of radius 4. Calculate the centroid of $\mathcal{W}$, i.e. the center of mass (COM) assuming that $\mathcal{W}$ has constant mass density.

Hint: Use the symmetry of the region to simplify your calculations.

W



In spherical coords we get for W:

$$W = \begin{cases} 0 \le \theta \le 2\pi \\ 0 \le \phi \le \frac{\pi}{4} \\ 0 \le r \le 4 \end{cases}$$

• By symmetry: $COM = (x_{com}, y_{com}, z_{com}) = \underline{\underline{(0, 0, z_{com})}}$

• $z_{com} = \dfrac{\iiint_W z \, dV}{\iiint_W 1 \, dV} = \dfrac{M_{xy}}{M}$

In spherical coords:

$$M = \int_{\theta=0}^{2\pi}\int_{\phi=0}^{\pi/4}\int_{r=0}^{4} r^2 \sin\phi \, dr \, d\phi \, d\theta = 2\pi \cdot \frac{r^3}{3}\Big|_{r=0}^{4} \cdot (-\cos\phi)\Big|_{\phi=0}^{\frac{\pi}{4}}$$

$$\underline{2\pi \cdot \frac{64}{3} \cdot \left(1 - \frac{\sqrt{2}}{2}\right)}$$

$$M_{xy} = \int_{\theta=0}^{2\pi}\int_{\phi=0}^{\pi/4}\int_{r=0}^{4} r^2 \sin\phi \cdot \underbrace{r \cos\phi}_{=z} \, dr \, d\phi \, d\theta$$

$$= 2\pi \cdot \frac{r^4}{4}\Big|_{r=0}^{4} \cdot \left(\frac{\sin^2\phi}{2}\Big|_{\phi=0}^{\frac{\pi}{4}}\right) = \underline{2\pi \cdot 64 \cdot \frac{1}{4} = 32\pi}$$

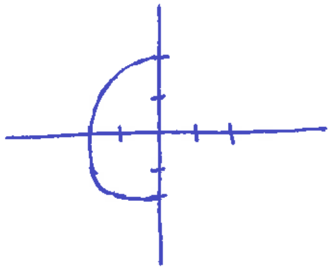
$$\Rightarrow z_{com} = \frac{\cancel{2\pi} \cdot \cancel{64} \cdot \frac{1}{4}}{\cancel{2\pi} \cdot \frac{\cancel{64}}{3} \cdot \left(1 - \frac{\sqrt{2}}{2}\right)} = \underline{\underline{\frac{3}{4\left(1 - \frac{\sqrt{2}}{2}\right)}}} = \underline{\underline{\frac{3}{2(2-\sqrt{2})}}}$$

(7) Let C be the half circle $x^2 + y^2 = 4$ with $x \le 0$.

(a) If C is oriented clockwise, and $\mathbf{F} = \langle -y, x \rangle$, what is $\int_C \mathbf{F} \cdot d\mathbf{r}$?

(b) Evaluate $\int_C e^{x^2+y^2}\, ds$.

(a)



$r(t) = (2\cos(t), 2\sin(t))$

$\frac{3\pi}{2} \le t \le \frac{\pi}{2}$ (orientation)

$r'(t) = (-2\sin(t), 2\cos(t))$

$$\int_C \bar{F} \cdot d\bar{r} = \int_{t_0}^{t_1} F(r(t)) \cdot r'(t)\, dt$$

$$= \int_{3\pi/2}^{\pi/2} (-2\sin(t), 2\cos(t)) \cdot (-2\sin(t), 2\cos(t))\, dt$$

$$= -\int_{\pi/2}^{3\pi/2} 4\sin^2(t) + 4\cos^2 t\, dt$$

$$= -\int_{\pi/2}^{3\pi/2} 4(\sin^2(t) + \cos^2(t))dt = -\int_{\pi/2}^{3\pi/2} 4\, dt = -\left[4t\right]_{\pi/2}^{3\pi/2} = -\left[4\left[3\pi/2 - \pi/2\right]\right]$$

$$= \boxed{-4\pi}$$

use parametrization from above (wit

(b) $$\int_C e^{x^2+y^2} ds = \int_{\pi/2}^{3\pi/2} e^{4\cos^2 t + 4\sin^2 t} \|(-2\sin t, 2\cos t)\|\, dt$$

$$= \int_{\pi/2}^{3\pi/2} e^4 \sqrt{4\sin^2 t + 4\cos^2 t}\, dt$$

$$= \int_{\pi/2}^{3\pi/2} e^4 \sqrt{4}\, dt = \int_{\pi/2}^{3\pi/2} 2e^4 dt = 2\pi e^4$$

$$\boxed{2\pi e^4}$$

(8) Let $\mathbf{F}$ be the vector field given by

$$\mathbf{F} = \left\langle yz^2\exp(xy) + 1\,,\ xz^2\exp(xy) + \frac{1}{z}\,,\ 2z\exp(xy) - \frac{y}{z^2} \right\rangle.$$

(a) Find a potential function for the vector field $\mathbf{F}$.

(b) Find the value of $\int_C \mathbf{F}\cdot d\mathbf{r}$, where C is the helix with parametrization

$$\mathbf{r}(t) = (2\cos t, 2\sin t, t+1), \quad \text{where} \quad 0 \le t \le 2\pi$$

(c) Let $\tilde{C}$ be the ellipse in the xy-plane with

$$\left(\frac{x}{2}\right)^2 + \left(\frac{y}{3}\right)^2 = 1 \qquad z = 0$$

What is the value of $\int_{\tilde{C}} \mathbf{F}\cdot d\mathbf{r}$? Explain your answer.

(a) $\dfrac{\partial f}{\partial x} = yz^2 e^{xy} \Rightarrow f = z^2 e^{xy} + g(y,z)$

$\dfrac{\partial f}{\partial y} = xz^2 e^{xy} + \dfrac{\partial g}{\partial y} \Rightarrow \dfrac{\partial g}{\partial y} = \dfrac{1}{z} \Rightarrow g = \dfrac{y}{z} + h(z)$

We now have $f = z^2 e^{xy} + \frac{y}{z} + h(z)$, and:

$\dfrac{\partial f}{\partial z} = 2ze^{xy} - \dfrac{y}{z^2} + \dfrac{dh}{dz} \Rightarrow \dfrac{dh}{dz} = 0,\ h(z) = C'.$

$$\boxed{f(x,y,z) = z^2 e^{xy} + \frac{y}{z} + C'}$$

We can take $C' = 0$.

(b) Since $\mathbb{F}$ is conservative: $\int_C \mathbb{F}\cdot d\bar{r} = f(Q) - f(P)$

with $P = r(0) = (2,0,1)$, $Q = r(2\pi) = (2,0,2\pi+1)$.

Then $f(2,0,2\pi+1) - f(2,0,1) = (2\pi+1)^2 - 1$
$= 4\pi^2 + 4\pi$

(c) $\tilde{C}$ is a closed curve, and $\mathbb{F}$ is conservative,

so therefore $\oint_{\tilde{C}} \mathbb{F}\cdot d\bar{r} = 0.$

(9) Consider the vector field

$$\mathbf{F}(x,y) = \left\langle \frac{x}{x^2+y^2}, \frac{y}{x^2+y^2} \right\rangle$$

defined on the domain $\mathcal{D} = \{(x,y) \neq (0,0)\}$.

(a) Show that $\mathbf{F}$ satisfies the cross-partials condition.
(b) Does the fact that $\mathbf{F}$ satisfies the cross partials condition guarantee that $\mathbf{F}$ is conservative? Explain why, or why not.
(c) $\mathbf{F}$ is conservative. Find a formula for its potential function $f(x,y)$.

(a) Let's check the cross partials

$$\frac{\partial F_1}{\partial y} = \frac{(x^2+y^2)(0) - x(2y)}{(x^2+y^2)^2} = \frac{-2xy}{(x^2+y^2)^2}$$

$$\frac{\partial F_2}{\partial x} = \frac{-2xy}{(x^2+y^2)^2}$$

$$\Rightarrow \boxed{\frac{\partial F_1}{\partial y} = \frac{\partial F_2}{\partial x}}$$

(b) It does not guarantee that F is conservative (Vice-versa is gauranteed). Here the domain D is not simply connected.

(c) We have

$$\frac{\partial f}{\partial x} = F_1 \Rightarrow f(x,y) = \int \frac{x}{x^2+y^2}\,dx + g(y)$$

$$\Rightarrow f(x,y) = \frac{1}{2}\ln|x^2+y^2| + g(y) \quad — (1)$$

$$\frac{\partial f}{\partial y} = F_2 \Rightarrow f(x,y) = \int \frac{y}{x^2+y^2}\,dy + h(x)$$

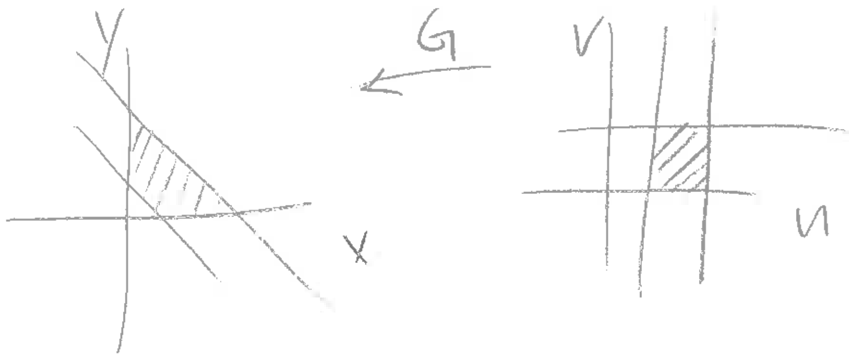
$$\Rightarrow f(x,y) = \frac{1}{2}\ln|x^2+y^2| + h(x) \quad — (2)$$

$$(1) \text{ \& } (2) \Rightarrow \boxed{f(x,y) = \frac{1}{2}\ln|x^2+y^2|}$$

(10) Let $\mathcal{D}$ be the region in the xy-plane bounded by the lines

$$x = 0, \quad y = 0, \quad x + y = 1, \quad x + y = 2$$

Evaluate the integral $\iint_{\mathcal{D}} xy\, dA$ by using a change of variables $x = u - uv$, $y = uv$, or in other words $G(u, v) = (u - uv, uv)$.



$x+y=1 \Rightarrow u-uv+uv = 1 \Rightarrow u=1$

$x+y=2 \Rightarrow u=2$ so $1 \leq u \leq 2$

$x=0 \Rightarrow u-uv=0 \Rightarrow u=0$ or $v=1$

$y=0 \Rightarrow uv=0 \Rightarrow u=0$ or $v=0$.

we know $1 \leq u \leq 2$, so we can ignore $u=0$.

then $0 \leq v \leq 1$

$$\text{Jacobian}(G) = \begin{vmatrix} 1-v & -u \\ v & u \end{vmatrix} = (1-v)u + uv = u$$

$1 \leq u \leq 2$ so $|u| = u$.

Therefore, $\iint_D xy\,dA = \int_0^1 \int_1^2 (u-uv)uv \cdot u \; du\,dv$

$$= \int_0^1 \int_1^2 u^3 (1-v) v \; du\,dv$$

$$= \frac{u^4}{4}\Big|_1^2 \cdot \left(\frac{v^2}{2} - \frac{v^3}{3}\right)\Big|_0^1$$

$$= \frac{15}{4} \cdot \frac{1}{6}$$

$$= \frac{5}{8}$$