Math 13 Winter 2015 Your name: ______________________________

Instructor (please circle): □ Pierre Clare □ Mingzhong Cai □ Erik van Erp

Problem set 2, due Wed Jan 21

Please show your work. No credit is given for solutions without justification.

(1) Find the volume of the tetrahedron bounded by the planes $z = 3y$, $x = 2y$, $x + 2y = 4$ and $z = 0$.

(2) Find the volume of the solid inside the sphere $x^2 + y^2 + z^2 = 16$ and *outside* the cylinder $x^2 + y^2 = 4$, and above the xy-plane $z = 0$.

(3) Find the center of mass of the half disk consisting of points in the plane with $x^2 + y^2 \leq 1$ and $y \geq 0$, assuming that the mass density is constant $\rho(x, y) = 1$.

① the volume

$$= \int_0^1 \int_{2y}^{4-2y} 3y \, dx \, dy$$

$$= \int_0^1 3y(x)\Big|_{x=2y}^{x=4-2y} dy$$

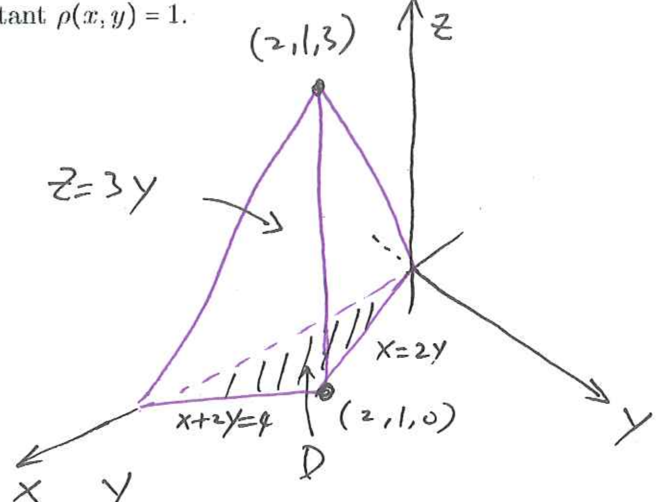
$$= \int_0^1 3y(4-4y) \, dy$$

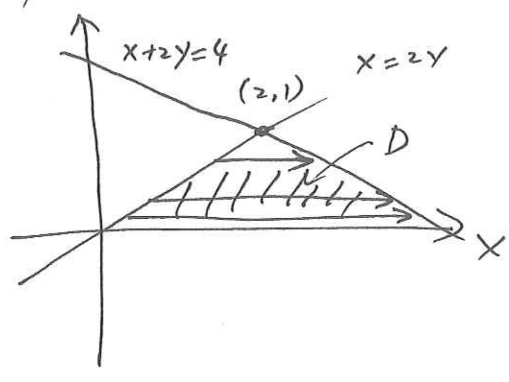
$$= 12 \int_0^1 y - y^2 \, dy$$

$$= 12 \left(\frac{1}{2}y - \frac{1}{3}y^3\right)\Big|_{y=0}^{y=1}$$

$$= 12\left(\frac{1}{2} - \frac{1}{3}\right)$$

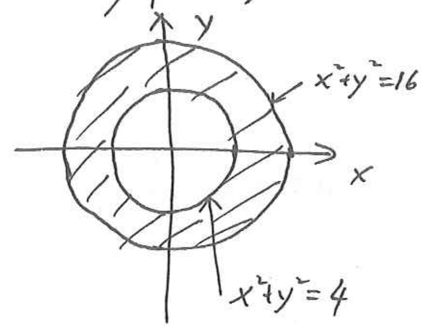
$$= 2$$





② $z=\sqrt{16-x^2-y^2}$ (since it is above the xy-plane)

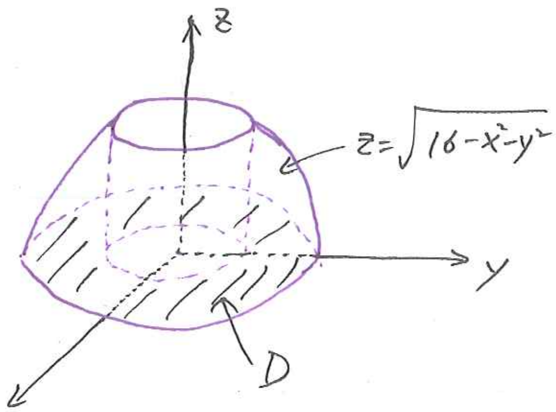
and the region D is as follows:



the volume of the solid =

$$\iint_D \sqrt{16-x^2-y^2}\, dA$$

$$= \int_0^{2\pi}\int_2^4 \sqrt{16-r^2}\, r\, dr\, d\theta$$



(Let $u=16-r^2$, $du=-2r\,dr$)

$$=\int_0^{2\pi} -\frac{1}{2} \times \frac{2}{3} u^{\frac{3}{2}}\Big|_{x=2}^{x=4} d\theta$$

$$=\int_0^{2\pi} -\frac{1}{3} \left(0^{\frac{3}{2}} - 12^{\frac{3}{2}}\right) d\theta$$

$$=2\pi\cdot\frac{1}{3}\cdot(2\sqrt{3})^3$$

$$=16\sqrt{3}\,\pi.$$

③ mass $= \iint_D 1\, dA = A(D) = \frac{\pi}{2}$

by symmetry $\bar{x} = 0$.

$$\bar{y} = \frac{1}{m}\cdot \iint_D y\, dA = \frac{2}{\pi}\int_0^{\pi}\int_0^1 r\sin\theta\, r\, dr\, d\theta$$

$$= \frac{2}{\pi}\cdot\int_0^{\pi} \frac{1}{3}\sin\theta\, d\theta = \frac{2}{\pi}\cdot\frac{1}{3}\cdot(-\cos\theta)\Big|_0^{\pi} = \frac{2}{\pi}\cdot\frac{1}{3}\cdot 2 = \frac{4}{3\pi}$$

So the center of mass is at $\left(0, \frac{4}{3\pi}\right)$.