Practice Final

1. Suppose $f(x, y) = (\ln(x^2 + y^2), e^{xy})$ and $g(s, t) = (s^2 e^t, (2s + t)^2)$. Find the matrix of the derivative $(f \circ g)'(1, 0)$.

2. Consider the parametrized surface
$$\mathbf{r}(u, v) = \langle 4u\cos(v), u^4, u\sin(v)\rangle.$$
   (a) Find the surface area of the part of the surface in the first octant and to the left of the plane $y = 1$ (so $y \le 1$).
   (b) Express the surface in Cartesian coordinates.
   (c) Find the equation of the tangent plane at the point $\mathbf{r}(1, 0)$. (You can use any method you prefer.)

3. Determine the volume of the solid bounded by $z = \sqrt{x^2 + y^2}$, $x^2 + y^2 + z^2 = 1$ and the xy-plane.

4. Determine the mass of a wire bent into the shape given by the equation $(x-1)^2+y^2 = 4$ if its linear mass density is equal to the square of its distance to the x-axis.

5. Let $L : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation satisfying $L(\mathbf{v}_1) = (5, 3)$ and $L(\mathbf{v}_2) = (3, 4)$, where $\mathbf{v_1} = (1, 1)$ and $\mathbf{v_2} = (0, 1)$.
   (a) Write down the representing matrix for L.
   (b) Let P be a parallelogram of area 3. If L maps P to the parallelogram Q, what is the area of Q?

6. Evaluate
$$\int_C 2xyz^2 dx \,+\, (x^2z^2 + z)\, dy \,+\, (2x^2yz + y)$$
where C is the curve
$$\mathbf{r}(t) = \langle e^{\sqrt{1-t^2}},\, t^3, \sqrt{3 + t^2}\rangle$$
$-1 \le t \le 1$ by any appropriate method.

7. Evaluate the following surface integrals. If you use any special techniques, be sure to justify that the technique is applicable.
   (a)
$$\int\int_S \mathbf{F} \cdot \mathbf{n}\, dS$$
where S is the positively oriented boundary (i.e., oriented with the outward normal) of the solid region lying above $z = \sqrt{\frac{x^2+y^2}{3}}$ and below $x^2 + y^2 + z^2 = 9$ and
$$\mathbf{F}(x, y, z) = \langle e^{y^2} + x, xz + 2y, z\rangle.$$

(b)

$$\int\int curl(\mathbf{F}) \cdot \mathbf{n}\, dS$$

where

$$\mathbf{F}(x, y, z) = \langle -y, x, e^{x^3+y^4} \rangle$$

and S is the hemisphere $x^2 + y^2 + z^2 = 4$, $z \geq 0$, oriented upwards. (S does not include the disk in the xy-plane.)

8. Suppose $\mathbf{F}$ can be written as the curl of another vector field $\mathbf{G}$. What is the value of $\iint_S \mathbf{F} \cdot d\mathbf{S}$, where S is the surface of a simple solid region in $\mathbb{R}^3$ with positive orientation? Justify your answer briefly.

9. Suppose that $\mathbf{F}$ is a vector field in $\mathbf{R}^3$ and that $f : \mathbf{R}^3 \to \mathbf{R}$ is a smooth function. For each of the following expressions, indicate whether the expression makes sense and, if so, whether it produces a vector field or a real-valued (scalar) function:

   (a) $Div(\nabla f)$
   (b) $\nabla(Div(\mathbf{F}))$
   (c) $curl(Div(\mathbf{F})$
   (d) $Div(curl(\mathbf{F}))$
   (e) $\nabla(curl(\mathbf{F}))$

10. Let E be a region in $\mathbf{R}^2$ bounded by a simple closed, piecewise smooth, positively oriented curve C. Which of the following is NOT equal to the area of E?

   (a) $\int_C -y\, dx$
   (b) $\int_C 2y\, dx + 3x\, dy$
   (c) $\int_C y\, dx + 2x\, dy$
   (d) $\int_C -y\, dx + x\, dy$
   (e) $\int_C x\, dy$

11. Indicate whether each of the following regions is simply-connected:

   (a) $\mathbf{R}^3$ with the origin removed.
   (b) $\mathbf{R}^3$ with the x axis removed.
   (c) $\mathbf{R}^3$ with the positive half of the x-axis removed.
   (d) $\mathbf{R}^3$ with the circle $x^2 + y^2 = 1$, $z = 0$, removed.

12. Find

$$\int_C e^{x^2}\,dx + x\,dy + xy\,dz$$

where C is the curve of intersection of the cylinder $x^2+y^2=1$ and the plane $x+2y+z=10$. Here C is assumed to be oriented counterclockwise when viewed from above.

13. Each of the following vector fields satisfies the condition $curl(\mathbf{F}) = 0$. In each case, determine whether the vector field is conservative. Justify your answer.

    (a) $\mathbf{F}(x,y) = \langle \frac{x}{x^2+y^2}, \frac{y}{x^2+y^2}, z^2 \rangle$

    (b) $\mathbf{F}(x,y) = \langle \frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2}, z^2 \rangle$