SURFACE AREA

Contents

1. Surface area

Consider the part of the surface $z = f(x, y)$ over the region D in the xy-plane. We saw how the double integral

$$\iint_D f(x, y)\, dA$$

represents the signed volume of the region between $z = f(x, y)$ and D. Another natural geometric question is to determine the *surface area* of this part of $z = f(x, y)$.

This problem is the higher dimensional analogue of the question of calculating the arc length of $y = f(x)$ over an interval $[a, b]$, so it should be no surprise that the formula will resemble the arc length formula. The surface area of $z = f(x, y)$ over a region D is equal to the integral

$$\iint_D \sqrt{1 + f_x(x, y)^2 + f_y(x, y)^2}\, dA.$$

We will not give an explanation of why this formula is true right now, as it will be a special case of a formula for surface area that we will encounter in about a month. Instead, we apply this formula to calculate the surface area of a few familiar objects.

Example. Find the surface area of $z = x^2 + y^2$ over the disc $x^2 + y^2 \leq 9$. Again, if $f(x, y) = x^2 + y^2$, then $f_x = 2x, f_y = 2y$, and so the surface area in question is

$$\iint_D \sqrt{1 + 4x^2 + 4y^2}\, dA = \int_0^{2\pi} \int_0^3 r\sqrt{1 + 4r^2}\, dr\, d\theta.$$

The iterated integral on the right hand side is equal to

$$2\pi \int_0^3 r\sqrt{1 + 4r^2}\, dr = 2\pi \left((1 + 4r^2)^{3/2} \cdot \frac{2}{3} \cdot \frac{1}{8} \Big|_{r=0}^{r=3} \right) = \frac{\pi}{6}(37^{3/2} - 1).$$

Example. Find the surface area of a sphere of radius a. We can let this sphere be $x^2 + y^2 + z^2 = a^2$. The top half of this sphere is given by $z = \sqrt{a^2 - x^2 - y^2} = f(x, y)$. The partial derivatives of f are

$$f_x(x,y) = \frac{-x}{\sqrt{a^2 - x^2 - y^2}}, f_y = \frac{-y}{\sqrt{a^2 - x^2 - y^2}},$$

so the surface area integral is

$$\iint\limits_D \sqrt{1 + \frac{x^2}{a^2 - x^2 - y^2} + \frac{y^2}{a^2 - x^2 - y^2}}\, dA = \iint\limits_D \sqrt{\frac{a^2}{a^2 - x^2 - y^2}}\, dA,$$

where D is the projection of the upper hemisphere onto the xy-plane. It is clear that D is described by polar inequalities $0 \leq r \leq 1, 0 \leq \theta \leq 2\pi$, so this double integral is equal to the iterated integral

$$\int_0^{2\pi} \int_0^a \frac{a}{\sqrt{a^2 - r^2}} r\, dr\, d\theta = 2\pi a \int_0^a \frac{r}{\sqrt{a^2 - r^2}}\, dA,$$

where we integrated with respect to θ first (which we can do since the integrand does not depend on θ) and pulled out a factor of a. The remaining expression equals

$$2\pi a(-\sqrt{a^2 - r^2}\Big|_{r=0}^{r=a}) = 2\pi a(a) = 2\pi a^2.$$

Notice this answer agrees with geometry, which tells us that the surface area of a sphere of radius a is $4\pi a^2$.

The prevalence of problems with polar coordinates is not an accident. Just like arc length, in general it is very difficult to calculate surface area, so problems have to be carefully chosen to be doable by hand. In many situations, the simplest problems to write down for calculating arc length are those which involve the use of polar coordinates.