APPLICATIONS OF MULTIPLE INTEGRATION

Contents

1. Physical interpretation of integrals

Having spent a considerable amount of time studying how to evaluate all sorts of different kinds of double integrals, we now briefly list some typical applications of double integration to physics and engineering. As a matter of fact, calculus was invented to allow scientists (such as Newton) the ability to calculate quantities from physics they were interested in.

Suppose we are given a *lamina*, which is a two-dimensional distribution of mass in some sort of shape. The shape will typically be specified by a domain D in the xy plane, and the distribution of mass might be specified by a density function $\rho(x, y)$ defined on D. Recall that the (average) density of a two-dimensional object with mass m and area A is given by m/A. The density function $\rho(x, y)$ can be interpreted as the limit of the densities of smaller and smaller squares around the point (x, y). (We ignore the issue of what units we use. As long as we are consistent with the choice of units for each of the quantities we are interested in, all the formulas below will work as is.)

Of course, in real life, objects are three-dimensional. Nevertheless, the simplification to two dimensions is instructive, because many of the calculations we perform will carry over to the three-dimensional case with little modification, yet be easier to do. Also, in some physical applications, it is sufficient to approximate a thin three-dimensional object as simply a two-dimensional object.

So suppose we have a lamina in the shape of a domain D with density function $\rho(x, y)$. A common question is to calculate the mass of the lamina. Because mass is equal to density times area, we can approximate the mass of a lamina with a Riemann sum

$$\sum \rho(x^*, y^*)\Delta x\,\Delta y$$

where we cut up the lamina into lots of small rectangles and then approximate the mass of each rectangle by multiplying the density at some point ($\rho(x^*, y^*)$ in the rectangle by the area of the rectangle ($\Delta x\,\Delta y$). As we take finer and finer approximations, the limit of these Riemann sums approaches the double integral

$$\iint_D \rho(x, y)\,dA.$$

This is the mass of the lamina, and we will often call it m.

Example. Suppose we have a lamina which is a triangle with vertices $(0,0),(0,1),(1,0)$, and density function $\rho(x,y)=y$. What is the mass of the lamina?

The formula for mass tells us that we should evaluate the double integral

$$\iint_D y\,dA.$$

The region D is defined by inequalities $0\le x\le 1, 0\le y\le 1-x$, so this double integral equals the iterated integral

$$\int_0^1\int_0^{1-x} y\,dy\,dx=\int_0^1 \frac{y^2}{2}\Big|_{y=0}^{y=1-x}dx=\int_0^1\frac{(1-x)^2}{2}\,dx=\frac{-(1-x)^3}{6}\Big|_0^1=\frac{1}{6}.$$

One of the interesting facts about physics is that different physical phenomena are governed by similar, or even the same, mathematical rules. For example, the forces of gravity and electricity act in very similar fashions – they both obey inverse square force laws. The discussion above could have worked just as well for a lamina of shape D with *charge distribution* $\rho(x,y)$, and then the double integral for mass above would instead calculate the total charge on the lamina. (One difference between gravity and electricity is that when dealing with mass, $\rho(x,y)\ge 0$, but when dealing with electric charge, ρ can be negative.) So even though we are calculating a different physical quantity, the mathematical calculations are identical.

Another quantity of interest associated to a lamina is its *center of mass*. This is the point at which the lamina can be balanced on a point, for example. Another interpretation of the center of mass of an object is that, when considering the force of gravity the object exerts on other objects, that force acts as if the entire mass of the lamina were concentrated at the center of mass. This was one of the problems calculus was initially created to solve – to simplify the calculation of gravitational forces exerted by various objects.

To help us calculate the center of mass of a lamina, we define the *moment about the x-axis* (respectively, moment about the y-axis) – this is not the moment of inertia! – to be the value of the integral

$$M_x=\iint_D y\rho(x,y)\,dA, M_y=\iint_D x\rho(x,y)\,dA.$$

Then the center of mass of the lamina has coordinates $\overline{x},\overline{y}$ given by

$$\overline{x}=\frac{M_y}{m},\overline{y}=\frac{M_x}{m}$$

Example. Consider a triangle with vertices at $(0,0),(3,0)$, and $(0,5)$, of uniform density (say $\rho(x,y)=1$). Where is the center of mass of this triangle located?

Let D be the triangle; it is determined by inequalities $0\le x\le 3, 0\le y\le 5-5x/3$. The mass of the triangle is easy to calculate in this situation; the triangle has area $3(5)/2=15/2=7.5$, so the mass is also $m=7.5$.

The moment of the triangle with respect to the x axis is given by

$$\int_0^3 \int_0^{5-5x/3} y\,dy\,dx = \int_0^3 \frac{y^2}{2}\Big|_{y=0}^{y=5-5x/3} dy = \int_0^3 \frac{(5-5x/3)^2}{2}\,dy.$$

This is equal to

$$\frac{(5-5x/3)^3}{6} \cdot \frac{-3}{5}\Big|_0^3 = \frac{5^3}{6} \cdot 35 = \frac{25}{2}.$$

Therefore the y coordinate of the center of mass is

$$\overline{y} = \frac{25}{2} \cdot \frac{2}{15} = \frac{5}{3}.$$

Similarly, the moment of the triangle with respect to the y axis is given by

$$\int_0^3 \int_0^{5-5x/3} x\,dy\,dx = \int_0^3 x(5-5x/3)\,dx = \frac{5x^2}{2} - \frac{5x^3}{9}\Big|_0^3 = \frac{45}{2} - 15 = \frac{15}{2}.$$

Therefore, the x coordinate of the center of mass is given by

$$\overline{x} = \frac{15}{2} \cdot \frac{2}{15} = 1.$$

Why such a funny triangle? We can 'experimentally' verify our calculations by taking a 3×5 index card, cutting it along a diagonal, and then try to balance the resulting triangle at the calculated center of mass. Sure enough, we find that mathematical theory agrees with real life!

Example. Let D be the half-annulus $9 \leq x^2 + y^2 \leq 16, y \geq 0$. Suppose we have a lamina whose shape is D and has uniform density. Find the center of mass.

The area of D is given by $16\pi - 9\pi/2 = 7\pi/2$, since we can take half the area of the difference of a circle of radius 4 and a circle of radius 3. (Of course, you could also evaluate a double integral, but the resulting calculation is longer than using ordinary geometry.) Since the region D involves circles, we will switch to polar coordinates. The polar inequalities which define D are $3 \leq r \leq 4, 0 \leq \theta \leq \pi$.

Symmetry strongly suggests that $\overline{x} = 0$, but we can calculate this quickly as well. We will assume that $\rho(x,y) = 1$. (We can do this because if $\rho(x,y) = c$, then there will be two factors of c which cancel when we calculate $\overline{x}, \overline{y}$.) The moment about the y-axis is given by

$$\iint\limits_D y\,dA = \int_0^\pi \int_3^4 r^2 \cos\theta\,dr\,d\theta.$$

If you evaluate the inner integral, you will get a constant times $\cos\theta$. Notice that the integral of $\cos\theta$ over $[0,\pi]$ is equal to 0. Therefore, $\overline{x} = 0$ as symmetry suggests.

The moment about the x-axis is given by

$$\iint\limits_D x\,dA = \int_0^\pi \int_3^4 r^2 \sin\theta\,dr\,d\theta = \int_0^\pi \frac{64}{3} - \frac{27}{3} \sin\theta\,d\theta = \int_0^\pi \frac{37}{3} \sin\theta\,d\theta = \frac{37}{3} \cdot 2 = \frac{74}{3}.$$

Therefore, the y coordinate of the center of mass is

$$\overline{y} = \frac{74}{3} \cdot \frac{2}{7\pi} = \frac{148}{21\pi} \approx 2.243.$$

(Beware! I had the wrong approximation in class!) Notice that the center of mass does not lie in D, and is actually below D. This principle is well-known to high jumpers, who arch their backs in a semi-circular fashion as they clear the bar. This allows them to have every part of their body above the bar while keeping the center of mass below the bar.

Another quantity of interest when studying rotational motion (torque, etc.) is the *moment of inertia*. For a point mass, the moment of inertia about a given axis ℓ of a point with mass m of distance r from ℓ is defined to be mr^2. Therefore, we define the moment of inertia of a lamina which occupies D and has density function $\rho(x,y)$ about the x-axis to be

$$I_x = \iint_D y^2 \rho(x,y)\,dA.$$

Similarly, the moment of inertia about the y-axis is

$$I_y = \iint_D x^2 \rho(x,y)\,dA.$$

The moment of inertia about the origin is given by

$$\iint_D (x^2+y^2)\rho(x,y)\,dA = I_x + I_y.$$

This can also be thought of as the moment of inertia about the z-axis, if we think of the lamina D as lying in the xy plane in $\mathbb{R}^3$.

Example. Wikipedia (or any introductory physics textbook) says that the moment of inertia of a disc about the x axis (or y axis, by symmetry) with radius R, centered at the origin, of uniform density, is given by $mR^2/4$. Show that this formula is correct.

Let $\rho(x,y) = 1$. Then the mass of the disc is given by $m = \pi R^2$. The moment of inertia about the x-axis is given by

$$\iint_D y^2\,dA.$$

Since D is circular, we use polar coordinates to evaluate this double integral. D is given by polar inequalities $0 \le r \le R, 0 \le \theta \le 2\pi$. Therefore, the iterated integral we want to calculate is

$$\int_0^{2\pi}\int_0^R (r\cos\theta)^2 r\,dr\,d\theta.$$

Evaluating the inner integral is easy: we get

$$\int_0^{2\pi} \frac{R^4}{4}\cos^2\theta\,d\theta.$$

The $R^4/4$ is a constant. We now need to integrate $\cos^2\theta$. This requires the use of the trigonometric identity

$$\cos^2\theta = \frac{1+\cos 2\theta}{2}.$$

If we use this, we obtain

$$\frac{R^4}{4} = \int_0^{2\pi} \frac{1+\cos 2\theta}{2}\,d\theta = \frac{R^4}{4}\left(\frac{\theta}{2} + \frac{\sin 2\theta}{4}\right)\Big|_0^{2\pi} = \frac{R^4}{4}(\pi + 0).$$

Since the mass $m = \pi R^2$, the moment about the x-axis can be rewritten as

$$\frac{mR^2}{4},$$

as desired.

As a matter of fact, you can now verify all sorts of formulas that you were given in physics class for centers of mass and moments of inertia using integration, and also calculate these quantities for a wider class of two-dimensional objects. One great feature of the mathematics behind the physics is that the mathematics is, by and large, the same, regardless of what quantities you are interested in calculating. For example, to compute the mass, center of mass, and moments of inertia of a lamina, you compute various integrals. Although the integrals are different, the techniques for calculating them are very similar, and you do not need to learn new mathematical ideas.

In a few weeks we will quickly see how to calculate these quantities for three-dimensional objects, or for that matter, n-dimensional objects (although $n = 3$ is by far the most common case, since it corresponds with the physical reality we observe) using triple or n-fold integrals.

2. Applications to Probability

A *continuous random variable* on $\mathbb{R}$ is a random variable X whose probability distribution is given by a function $f : \mathbb{R} \to \mathbb{R}$ satisfying

$$\int_{-\infty}^{\infty} f(x)\,dx = 1.$$

You can think of X as being the outcome of a measurement of some sort of random process, where the outcome is some real number, and the probability that the measurement is between a, b is given by

$$\int_a^b f(x)\,dx.$$

Continuous random variables appear throughout physics, engineering, economics, and finance. For example, they can model the distribution of many different types of random processes; for example, the number of times a geiger counter ticks in some length of time, or the number of searches typed into Google during some time interval, etc. (Technically speaking, these distributions are not continuous, in that X only takes some finite number of values, but oftentimes they are modeled by continuous random variables because the range of possible output values is so large.)

In quantum mechanics, one learns that particles behave according to a probability distribution called a wave function, which governs the probability distribution of results of experiments done on that particle.

This is not a class in probability, so we will only cover some of the most basic types of random variables. For example, a uniformly distributed random variable on an interval $[a, b]$ is the random variable given by distribution

$$f(x) = \begin{cases} \frac{1}{b-a} & \text{if } a \leq x \leq b, \\ 0 & \text{otherwise.} \end{cases}$$

This random variable takes values between a, b, with every value in this interval being taken with equal probability.

We can use multiple integrals to describe probability distributions of two or more random variables. For example, suppose X, Y are two random variables. We say that $f(x, y)$ is their joint probability distribution if

$$\iint_{\mathbb{R}^2} f(x, y)\, dA = 1,$$

where we interpret

$$\int_a^b \int_c^d f(x, y)\, dy\, dx$$

to be the probability that $a \leq X \leq b$ and $c \leq Y \leq d$. There are many different types of simple (and not so simple) questions about joint probability distributions that can be answered using multiple integration.

Example. Suppose X, Y are independent uniformly distributed random variables on $[0, 1], [0, 2]$ respectively. Find the probability that $X \leq 1/2, Y \geq 1$ First, independent means, roughly speaking, that the value of X does not depend on the value of Y, and vice versa. Mathematically speaking, if X, Y are independent with distribution $f(x), g(x)$, respectively, then their joint probability distribution is given by $f(x)g(y)$. (It is a good exercise to check that this is indeed a joint probability distribution, and that the value of X is independent of the value of Y.)

In any case, returning to the problem at hand, we just use the definition of what a joint probability distribution is. First, $f(x) = 1$ on $[0, 1]$ is the probability distribution for X, and $g(y) = 1/2$ on $[0, 2]$ is the distribution for Y, so $h(x, y) = 1/2$ on $[0, 1] \times [0, 2]$ is the joint distribution for X, Y. Then the probability in question is just

$$\int_1^2 \int_0^{1/2} h(x, y)\, dA = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}.$$

Example. Suppose that X, Y are independent uniformly distributed random variables on $[0, 1]$. What is the probability that $X + Y \leq 1/2$? Again, first we note that $f(x) = 1, g(y) = 1$ on $[0, 1]$ are the distributions for these values, so $h(x, y) = 1$ on $[0, 1] \times [0, 1]$ is the joint distribution for X, Y. To determine the probability that $X + Y \leq 1/2$, we want to integrate over the triangular region $0 \leq x \leq 1/2, 0 \leq y \leq$

$1/2 - x$. Indeed, it is exactly points in this region that correspond to measurements where $X + Y \leq 1/2$. Therefore the answer is

$$\int_0^{1/2} \int_0^{1/2-x} 1 \, dA = \frac{1}{8}.$$

Another way to approach this problem is to determine what the probability distribution for $X + Y$ is (assuming that X, Y are independent), in terms of $f(x), g(y)$. Let us call the corresponding probability distribution for $X + Y$ $u(x)$. Then

$$\int_a^b u(x) \, dx$$

tells us the probability that $a \leq X + Y \leq b$. We know the joint distribution is given by $f(x)g(y)$. Therefore this probability equals

$$\int_{-\infty}^{\infty} \int_{a-y}^{b-y} f(x)g(y) \, dx \, dy,$$

because the region we are integrating over corresponds exactly to the region where $a \leq X + Y \leq b$. If we make the substitution $x \to x - y$ on the inner integral, we get

$$\int_{-\infty}^{\infty} \int_a^b f(x-y)g(y) \, dx \, dy.$$

Because the bounds of integration are now constants, we can interchange the order of integration, which gives

$$\int_a^b \int_{-\infty}^{\infty} f(x-y)g(y) \, dy \, dx.$$

This implies that the function

$$(f * g)(x) = \int_{-\infty}^{\infty} f(x-t)g(t) \, dt = \int_{-\infty}^{\infty} f(t)g(x-t) \, dt$$

is the probability distribution for $X + Y$.

This function is known as the *convolution* of f and g, and is of central importance in many different branches of mathematics, physics, and engineering.