INTEGRATION IN POLAR COORDINATES

Contents

1. A review of polar coordinates

There are many situations where we may want to integrate a function over a circular or elliptical domain. In this situation, describing the upper and lower boundary of the domain involves functions like $\sqrt{1-x^2}$, which may be hard to work with. Instead, we might instead ask if it is possible to calculate these integrals with respect to *polar coordinates*, which are a more natural coordinate system to use when dealing with circles, ellipses, and more elaborate curves, such as cycloids.

Before discussing integration with polar coordinates, we will review the concept of polar coordinates. Instead of describing the position of a point in the xy plane by using its distance from the x and y axes, we can describe a point by giving its distance from the origin, and the angle the segment connecting the point with the origin makes with the positive x-axis. This is somewhat akin to how points might be described in a radar system.

More specifically, if we say that a point in the xy-plane has polar coordinates (r, θ), then we are talking about the point (x, y) with rectangular coordinates

$$x = r\cos\theta, y = r\sin\theta.$$

One complication that arises when using polar coordinates is the fact that the polar coordinates for a given point are not unique. For example, consider the point with polar coordinates $(1, \pi/2)$. This is the point $(0, 1)$ in rectangular coordinates. However, the points with polar coordinates $(-1, -\pi/2), (1, 5\pi/2), (1, 9\pi/2), \ldots$ all represent this point as well. In general, the point (r, θ) can also be represented by $(r, \theta + 2n\pi)$, where n is any integer – this represents the fact that we can go 2π radians around a circle and return to the same point, and also by $(-r, \theta + \pi)$, which represents the fact that going around a circle by π radians is the same as reflecting a point through the origin.

We want equations which let us convert from rectangular to polar coordinates and vice versa. Given a point (x, y) in rectangular coordinates, we can let $r = \sqrt{x^2 + y^2}$. This guarantees that r is always non-negative and also determines r uniquely. The formula for θ is not quite as nice. One possibility is

$$\theta = \arctan(y/x) \text{ if } x > 0, \theta = \arctan(y/x) + \pi \text{ if } x < 0$$

up to the ambiguity by a multiple of 2π. However, in general, if you need to convert rectangular to polar coordinates by hand, it is frequently easier to just draw a sketch of the point (x, y) and then calculate θ from trigonometric principles.

Converting from polar to rectangular coordinates is much easier. If a point has polar coordinates (r, θ), then its rectangular coordinates are given by the formulas $x = r\cos\theta, y = r\sin\theta$ listed above.

Examples.

- Polar coordinates are very well suited to working with circles. For example, a circle with equation $x^2 + y^2 = 1$ is defined by the simple polar equation $r = 1$.
- The equation $r = \theta$ defines a spiral, sometimes known as the *spiral of Archimedes*.
- The polar equation $\theta = \pi/2$ is the same as the line $y = 0$.

2. Integrating using polar coordinates

Suppose we want to calculate the volume of a sphere by taking a double integral. For example, consider the sphere $x^2 + y^2 + z^2 = 1$; to calculate this volume using an integral we will want to calculate an integral of the form

$$\iint\limits_D \sqrt{1 - x^2 - y^2}\, dA$$

where D is the disc $x^2 + y^2 \le 1$. (This integral is actually equal to half the volume of the unit sphere.) As an iterated integral, this double integral is equal to

$$\int_{-1}^{1}\int_{0}^{\sqrt{1-x^2}} \sqrt{1 - x^2 - y^2}\, dy\, dx.$$

It is possible to evaluate this integral, although it is not entirely pleasant. How can we use polar coordinates to simplify the calculation of this integral?

We should start by converting the function being integrated, which in this case is, $f(x, y) = \sqrt{1 - x^2 - y^2}$, to polar coordinates, by using the substitution $x = r\cos\theta, y = r\sin\theta$. We see that the result is

$$f(r\cos\theta, r\sin\theta) = \sqrt{1 - r^2}.$$

(We use the fact that $\cos^2\theta + \sin^2\theta = 1$, which is a basic identity to know when working with polar coordinates or performing trigonometric integrals.) We also need to convert the bounds of integration, which indicate the domain we are integrating over, from rectangular to polar coordinates. When doing this it is often best to draw a sketch.

In this example, we are integrating over the unit disc $x^2 + y^2 \le 1$. This is evidently the same region as that defined by polar inequalities $0 \le r \le 1, 0 \le \theta \le 2\pi$. Notice that we need to restrict θ so that every point of the disc, except possibly those which lie on the positive x-axis, is represented only once by polar coordinates.

It turns out that to calculate an integral using polar coordinates, not only do we need to convert the function being integrated from rectangular to polar coordinates as well as the region of integration, but we also need to insert an additional factor of r in the integrand. More specifically, if f is continuous on a region D defined by polar coordinates $0 \le a \le r \le b, \alpha \le \theta \le \beta$, where $0 \le \beta - \alpha \le 2\pi$, then

$$\iint_D f(x,y)\,dA = \int_\alpha^\beta \int_a^b f(r\cos\theta, r\sin\theta) r\,dr\,d\theta.$$

This extra factor of r might seem somewhat mysterious, but it is analogous to how when performing a u-substitution in integration, we have a factor of $u'(x)$ which appears:

$$\int f(u)\,du = \int f(u(x))u'(x)\,dx.$$

We are carrying out the higher dimensional analogue of u-substitution using the specific change of variables $x = r\cos\theta, y = r\sin\theta$, and the factor of r appearing in the integrand plays the role of $u'(x)$. We will see how this fits into a more general formula for change of variables in a few weeks.

In our example, converting the integral from rectangular to polar coordinates yields

$$\int_0^{2\pi} \int_0^1 \sqrt{1-r^2}\, r\,dr\,d\theta.$$

Notice that the extra factor of r makes the resulting integral easy to compute by u-substitution! We have

$$\int_0^{2\pi} \frac{(1-r^2)^{3/2}}{-2} \cdot \frac{2}{3}\Big|_0^1 d\theta = \int_0^{2\pi} \frac{1}{3} d\theta = \frac{2\pi}{3}.$$

This accords with what we know to be true from geometry, which tells us that a sphere of radius 1 has volume $4\pi/3$.

Example. Use polar coordinates to evaluate the double integral

$$\iint_D 1 - \sqrt{x^2+y^2}\,dA$$

where D is the unit disc $x^2 + y^2 \le 1$. To the volume of what geometric figure does this double integral equal? Use this to check that your answer is correct.

We first draw a sketch of the geometric figure whose volume this double integral represents. The graph of $z = 1 - \sqrt{x^2+y^2}$ can be thought of as the graph of $z = 1 - r$, where we convert rectangular coordinates in the xy plane to polar coordinates. But this is just a cone whose base is the disc D and has height 1.

Again, notice that while it is possible to evaluate this integral in rectangular coordinates using trigonometric substitutions, it certainly seems like it is (and will be) easier to carry out this integral using polar coordinates. The unit disc D is given by polar inequalities $0 \le r \le 1, 0 \le \theta \le 2\pi$, and we have already seen that the integrand can be expressed as $1 - r$. Therefore, the iterated integral we want to calculate is

$$\int_0^{2\pi} \int_0^1 (1-r) r\,dr\,d\theta.$$

(Remember that additional factor of r!) This is easy to calculate; we get

$$\int_0^{2\pi} \left(\frac{r^2}{2} - \frac{r^3}{3} \Big|_0^1 \right) d\theta = \int_0^{2\pi} \frac{1}{6} \, d\theta = \frac{\pi}{3}.$$

This agrees with what geometry tells us, since a cone whose base has area π and has height 1 will have volume $\pi/3$.

So far, we have discussed evaluating integrals over domains which are polar rectangles; i.e. of the form $a \leq r \leq b, \alpha \leq \theta \leq \beta$. However, we can also consider more general domains of the form $r_1(\theta) \leq r \leq r_2(\theta), \alpha \leq \theta \leq \beta$, where the inequalities for r depend on θ. And just like the case with rectangular coordinates, the corresponding iterated integral becomes

$$\int_\alpha^\beta \int_{r_1(\theta)}^{r_2(\theta)} f(r\cos\theta, r\sin\theta) r \, dr \, d\theta.$$

Example. Find the area of the region defined by the polar inequalities $0 \leq \theta \leq 2\pi, 0 \leq r \leq \theta$. (Notice that this is the area 'enclosed' by one loop of the spiral of Archimedes.)

If we let D be the region defined by the above inequalities, then the area of D is given by the iterated integral

$$\iint_D dA = \int_0^{2\pi} \int_0^\theta r \, dr \, d\theta.$$

Evaluating this iterated integral gives:

$$\int_0^{2\pi} \left(\frac{r^2}{2} \Big|_{r=0}^{r=\theta} \right) d\theta = \int_0^{2\pi} \frac{\theta^2}{2} \, d\theta = \frac{\theta^3}{6} \Big|_0^{2\pi} = \frac{4\pi^3}{3}.$$

Notice that this integral probably would have been very hard to calculate using rectangular coordinates.

Example. Let D be the half-annulus given by the inequalities $1 \leq x^2 + y^2 \leq 4, y \geq 0$. Evaluate the double integral

$$\iint_D \sin(x^2 + y^2) \, dA.$$

By now, it should be fairly clear that a good place to start with this problem is to convert this integral into an iterated integral over polar coordinates, since the domain of integration seems well suited to the use of polar coordinates. (If you try to evaluate this integral using rectangular coordinates you will fail, anyway, since you end up needing to evaluate the integral of $\sin x^2$.)

The region D is described by polar inequalities $1 \leq r \leq 2, 0 \leq \theta \leq \pi$. Therefore, the iterated integral we want to calculate is

$$\int_0^\pi \int_1^2 \sin((r\cos\theta)^2 + (r\sin\theta)^2) r \, dr \, d\theta = \int_0^\pi \int_1^2 r \sin(r^2) \, dr \, d\theta.$$

Notice that the factor of r saves the day again! We evaluate this integral:

$$\int_0^{\pi} \left(\frac{-\cos(r^2)}{2} \bigg|_{r=1}^{r=2} \right) d\theta = \int_0^{\pi} \frac{\cos 1 - \cos 4}{2}\, d\theta = \frac{\pi(\cos 1 - \cos 4)}{2}.$$

Example. We now consider a very clever use of polar coordinates to evaluate an important integral. Consider the improper integral

$$I = \int_{-\infty}^{\infty} e^{-x^2}\, dx.$$

We can't evaluate this integral by finding an antiderivative for the integrand since there is no such expression for this antiderivative using any functions we know. Nevertheless, it is possible to calculate the value of this definite integral using integration over polar coordinates and a very clever trick!

This integral is of central importance in probability theory, because it turns out to be very closely related to the standard normal distribution, which is also known as the Gaussian or Bell curve. Therefore, it is very important to know how to calculate the value of this integral.

Since there don't seem to be any double integrals in sight, we manufacture one in the following clever way. Consider I^2, written in the following strange form:

$$I^2 = \left(\int_{-\infty}^{\infty} e^{-x^2}\, dx \right) \left(\int_{-\infty}^{\infty} e^{-y^2}\, dy \right).$$

The clever observation is that this product of integrals can be rewritten as a single double integral, over the domain $\mathbb{R}^2$. (We technically have not defined double integrals over unbounded regions, but the idea is similar to how improper integrals are defined. One takes a limit of the value of this double integral taken over larger and larger rectangles.) It is a general fact that the double integral of a function of the form $f(x)g(y)$ over the rectangle $R = [a, b] \times [c, d]$ is equal to

$$\int_a^b \int_c^d f(x)g(y)\, dy\, dx = \left(\int_a^b f(x)\, dx \right) \left(\int_c^d g(y)\, dy \right).$$

You can see this just by evaluating the iterated integral on the left. In any case, using this fact, we can rewrite I^2 as follows:

$$I^2 = \iint_{\mathbb{R}^2} e^{-x^2} e^{-y^2}\, dA = \iint_{\mathbb{R}^2} e^{-(x^2+y^2)}\, dA.$$

At this point, we write this double integral as an iterated integral using polar coordinates. (One might think to do this because of the appearance of $x^2 + y^2$ in the integrand.) First, the domain $D = \mathbb{R}^2$ can be expressed using polar coordinates as $0 \le r \le \infty, 0 \le \theta \le 2\pi$. Therefore, the integral we want to evaluate is

$$\int_0^{2\pi} \int_0^{\infty} r e^{-r^2}\, dr\, d\theta.$$

Notice that the extra factor of r allows us to evaluate this integral:

$$\int_0^{2\pi} \left(\frac{-e^{-r^2}}{2} \Bigg|_{r=0}^{\infty} \right) d\theta = \int_0^{2\pi} \frac{1}{2} d\theta = \frac{2\pi}{2} = \pi.$$

Therefore, the value of the integral which we wanted to originally calculate is $I = \sqrt{\pi}$.