VECTOR-VALUED FUNCTIONS, ARC LENGTH, FUNCTIONS OF SEVERAL VARIABLES

Contents

Partial derivatives not only tell us the rate of change of a function $f(x, y)$ in either the x or y direction, they can also help us calculate objects of interest such as tangent planes and normal lines. Furthermore, they let us calculate the gradient of f, which allows us to easily compute directional derivatives, among other things.

1. Directional derivatives and the gradient

Let $f(x, y)$ be a function of two variables. Then the partial derivatives f_x, f_y can be interpreted as the rate of change of a function in either the x or y direction. However, there is nothing intrinsically special about these two directions: we may just as well ask what the rate of change of $f(x, y)$ in some other direction is.

More specifically, suppose we want to study the rate of change of $f(x, y)$ at a point (a, b). A direction can be specified by giving a unit vector; this vector points in some direction, and is uniquely determined by a direction. Let $\mathbf{u}$ be such a vector; we sometimes call unit vectors in this context a *direction vector*. Then we can ask how $f(x, y)$ changes at (a, b) as we go in the direction of $\mathbf{u}$. We define the *directional derivative* of $f(x, y)$ at $\mathbf{v} = (a, b)$ in the direction of the unit vector $\mathbf{u}$ to be the value of the limit

$$D_{\mathbf{u}} f(a, b) = \lim_{h \to 0} \frac{f(\mathbf{v} + h\mathbf{u}) - f(\mathbf{v})}{h}.$$

Partial derivatives are given by either letting $\mathbf{u} = \langle 1, 0 \rangle$ or $\langle 0, 1 \rangle$. Intuitively, if we think of $f(x, y)$ as the height of a hill, then the directional derivative at (a, b) in the direction of $\mathbf{u}$ is the rate at which the height increases or decreases if we walk in the direction of $\mathbf{u}$ at (a, b).

How can we quickly calculate directional derivatives? To answer this question, we introduce the *gradient* of a function $f(x, y)$. The gradient of $f(x, y)$, written $\nabla f(x, y)$, is defined to be the function

$$\nabla f(x, y) = \langle f_x(x, y), f_y(x, y) \rangle.$$

This is a function which has domain $\mathbb{R}^2$, and takes values in $\mathbb{R}^2$: that is, the gradient of f is a vector-valued function defined on $\mathbb{R}^2$.

It turns out that directional derivatives can easily be calculated in terms of $\nabla f(x, y)$:

$$D_{\mathbf{u}} f(a, b) = \nabla f(a, b) \cdot \mathbf{u}.$$

When using this formula to calculate partial derivatives, be absolutely sure that you are using a unit vector for $\mathbf{u}$.

Example. Calculate the directional derivative of $f(x, y) = x^2 + y^2$ at $(4, 7)$ in the direction $\langle 1, 2\rangle$. Remember that when calculating directional derivatives, our directions need to be specified by a unit vector. The unit vector that points in the same direction as $\langle 1, 2\rangle$ is $\langle 1/\sqrt{5}, 2/\sqrt{5}\rangle$. The gradient of $f(x, y)$ is $\nabla f = \langle 2x, 2y\rangle$. In particular, $\nabla f(4, 7) = \langle 8, 14\rangle$. Then the directional derivative in question is

$$\langle 8, 14\rangle \cdot \frac{1}{\sqrt{5}}\langle 1, 2\rangle = \frac{36}{\sqrt{5}}.$$

This formula also allows us to easily see two properties of the gradient vector. Since $|\mathbf{u}| = 1$ regardless of the choice of $\mathbf{u}$, the directional derivative $D_{\mathbf{u}}f(a, b)$ is evidently maximized when $\mathbf{u}$ points in the same direction as $\nabla f(a, b)$, because

$$\nabla f(a, b) \cdot \mathbf{u} = |\nabla f(a, b)||\mathbf{u}| \cos\theta = |\nabla f(a, b)| \cos\theta$$

which is clearly maximized when $\theta = 0$. Furthermore, the rate of change in the direction of maximum increase is given by $|\nabla f(a, b)|$. Therefore, we see that the gradient vector (1) points in the direction in which a function is increasing most rapidly, and (2) the magnitude of the gradient vector tells us the rate of this increase.

Example. Consider $z = x^2 + y^2$. At the point $(2, 3)$, in what direction is z increasing most rapidly? How rapidly is z increasing in that direction? We begin by calculating $\nabla z = \langle 2x, 2y\rangle$. Therefore, $\nabla z(2, 3) = \langle 4, 6\rangle$. This is the direction in which z is increasing most rapidly. Furthermore, z is increasing at a rate of $|\nabla z(2, 3)| = \sqrt{4^2 + 6^2} = 2\sqrt{13}$ in this direction.

2. Tangent planes and normal lines

Recall that the derivative of a single variable function $f(x)$ can be interpreted as the slope of the tangent line to the graph $y = f(x)$. In particular, at x_0, this tangent line has equation

$$y - f(x_0) = f'(x_0)(x - x_0).$$

We seek a similar formula for the tangent plane to a graph of a function of two variables. Intuitively, it is somewhat clear that a surface should usually have many lines tangent to it, and perhaps less obvious that these lines will form a plane. It turns out that if a function $f(x, y)$ is differentiable at a point (x_0, y_0), then the tangent plane is given by the equation

$$z - f(x_0, y_0) = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0).$$

This formula is very similar to the equation for a tangent line.

Example. Find the tangent plane to $f(x, y) = xy + y^2$ at $(1, 2)$. We find $f_x = y, f_y = x + 2y$, so $f_x(1, 2) = 2, f_y(1, 2) = 5$. The equation for the tangent plane is thus

$$z - 6 = 2(x - 1) + 5(y - 2).$$

There is actually a more general equation for the tangent plane to a surface given by an equation of the form $F(x, y, z) = C$, for some constant C. (In particular, the case of a graph of a function $f(x, y)$ is the special case $F(x, y, z) = z - f(x, y) = 0$.) The tangent plane to $F(x, y, z) = C$ at (x_0, y_0, z_0) is given by the formula

$$F_x(x_0, y_0, z_0)(x - x_0) + F_y(x_0, y_0, z_0)(y - y_0) + F_z(x_0, y_0, z_0)(z - z_0) = 0.$$

In particular, notice that $\nabla F(x_0, y_0, z_0)$ is a normal vector for this plane. This tells us that another interpretation of ∇F is as a vector which is orthogonal to level curves/surfaces of F.

Finally, this formula also provides us with a convenient way to calculate the *normal line* to a surface, which are the lines which are orthogonal to the tangent planes of a surface. Since $\nabla F(x_0, y_0, z_0)$ is normal to a tangent plane, a vector equation for a normal line is given by

$$\langle x_0, y_0, z_0\rangle + t\langle F_x(x_0, y_0, z_0) + F_y(x_0, y_0, z_0) + F_z(x_0, y_0, z_0)\rangle.$$

Of course, this discussion of normal lines is valid not only for functions $F(x, y, z) = C$, but also $F(x, y) = C$.

Examples.

- Determine the equations for the normal lines to the graph of $x^2 - y^2 = 1$ for a general point on this graph. In this situation, $F(x, y) = x^2 - y^2$, so $\nabla F(x, y) = \langle 2x, -2y\rangle$. Therefore, the normal line at x_0, y_0 is given by

$$\langle x_0, y_0\rangle + t\langle 2x_0, -2y_0\rangle.$$

- Consider the sphere $x^2 + y^2 + z^2 = 9$. Calculate the equation for the tangent plane and normal line to the sphere at $(2, 1, 2)$.

  We begin by calculating the gradient of $f(x, y, z) = x^2 + y^2 + z^2$. We see that $\nabla f = \langle 2x, 2y, 2z\rangle$. Therefore, the gradient at $(2, 1, 2)$ is equal to $\nabla f(2, 1, 2) = \langle 4, 2, 4\rangle$. Therefore, the tangent plane to $x^2 + y^2 + z^2 = 9$ at $(2, 1, 2)$ has normal vector $\langle 4, 2, 4\rangle$. The equation of this plane must then be

$$4x + 2y + 4z = 18, \text{or } 2x + y + 2z = 9.$$

  The normal line has direction vector $\langle 4, 2, 4\rangle$ and passes through $(2, 1, 2)$. Therefore, the normal line is given by parametric equations $x = 2 + 4t, y = 1 + 2t, z = 2 + 4t$. Notice that this line passes through the origin.