THE DIVERGENCE AND CURL

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We've spent a considerable amount of time discussing line integrals, and in particular, line integrals of conservative vector fields. While much of what we learned is valid for line integrals over curves in $\mathbb{R}^n$, where n is arbitrary, some of it is specific for $\mathbb{R}^2$. In particular, we saw that if a vector field $\mathbf{F} = \langle P, Q \rangle$ is conservative, then $P_y = Q_x$, and the converse statement holds if the vector field is defined on a simply-connected set. However, there is no obvious generalization of this condition to a vector field $\mathbf{F} = \langle P, Q, R \rangle$ in $\mathbb{R}^3$.

It is partially for this reason that we will study two operators, defined on vector fields in $\mathbb{R}^3$, called the *divergence* and *curl*. However, they turn out to be important for many other reasons, both in science and in mathematics, because they appear in mathematical descriptions of natural phenomena.

1. The divergence

Let $\mathbf{F} = \langle P, Q, R \rangle$ be a vector field in $\mathbb{R}^3$. Then the *divergence* of $\mathbf{F}$, written div $\mathbf{F}$ or $\nabla \cdot \mathbf{F}$, is defined to be the scalar function on $\mathbb{R}^3$ given by

$$\nabla \cdot \mathbf{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}.$$

Actually, we could have defined the divergence of a vector field $\mathbf{F} = \langle F_1, \ldots, F_n \rangle$ on $\mathbb{R}^n$ for any n, not just $n = 3$, as the sum of partial derivatives

$$\nabla \cdot \mathbf{F} = \frac{\partial F_1}{\partial x_1} + \frac{\partial F_2}{\partial x_2} + \ldots + \frac{\partial F_n}{\partial x_n},$$

but we will only be concerned with the cases $n = 2, 3$ in this class. We sometimes say 'del dot $\mathbf{F}$' instead of the divergence of $\mathbf{F}$, which is supposed to recall the $\nabla \cdot \mathbf{F}$ notation.

You can only calculate the divergence of a vector field, not a scalar function, and the divergence of a vector field is a scalar function. The notation $\nabla \cdot \mathbf{F}$ is supposed to remind you of the gradient of a scalar function, although the divergence is something related to, but distinct from, the gradient.

As a matter of fact, the notation $\nabla \cdot \mathbf{F}$ can be thought of as a mnemonic device for remembering the definition of divergence. If you pretend that the symbol ∇ represents the 'vector'

$$\nabla = \langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \rangle,$$

then the 'dot product' of ∇ with $\mathbf{F} = \langle P, Q, R \rangle$ is given by

$$\nabla \cdot \mathbf{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}.$$

Of course, ∇ really isn't a vector, since vectors must have numbers as coordinates, but we can use this interpretation of ∇ and $\nabla \cdot \mathbf{F}$ to help us remember the definition of divergence.

In practice, the divergence is usually easy to calculate:

Examples.

- Calculate the divergence of $\mathbf{F} = \langle 2x + y, \cos y \rangle$. Since $P = 2x + y, Q = \cos y$, and $P_x = 2, Q_y = -\sin y$, we have $\nabla \cdot \mathbf{F} = 2 - \sin y$.
- Calculate the divergence of $\mathbf{F} = \langle xe^z, x+y+z, 3y+z^2 \rangle$. Again, $P = xe^z, Q = x + y + z, R = 3y + z^2$, so $P_x = e^z, Q_y = 1, R_z = 2z$, and therefore $\nabla \cdot \mathbf{F} = e^z + 1 + 2z$.

These examples should convince you that calculating the divergence is pretty easy (certainly about the same difficulty as calculating a gradient), but why bother even calculating the divergence of a field anyway? For example, we know that the derivative of a function $f(x)$ represents the rate of change of a function, while the gradient of a function $f(x, y)$ tells us the direction that function increases most rapidly and how rapidly it increases in that direction. What does the divergence tell us?

Let's suppose we have a vector field $\mathbf{F}$. If we pretend that $\mathbf{F}$ represents the rate of flow of a fluid, for example, the $\nabla \cdot \mathbf{F}$ turns out to represent the amount of 'accumulation' of that fluid at a given point. For example, if $\nabla \cdot \mathbf{F}$ is positive at a point, then this means that more of the fluid is flowing out of the point than into the point, and we sometimes call such a point a *source*. On the other hand, if $\nabla \cdot \mathbf{F}$ is negative, this means more of the fluid is flowing in than out, and we sometimes call such a point a *sink*. This is all best illustrated by looking at a variety of examples where we can easily see whether a flow is accumulating or not, at least intuitively.

We can formulate a more precise (but not entirely precise, at least yet) version of this intuitive idea. Suppose we are interested in $\nabla \cdot \mathbf{F}$ at a point (x, y). (The same idea works in three dimensions.) Draw a small box D around (x, y). If we think of $\mathbf{F}$ as a fluid, and measure the total amount of fluid leaving the box and subtract that from the amount entering, we get a number, say l_D. We divide this number by the area of D, and take the limit as D becomes arbitrarily small, and this turns out to equal $\nabla \cdot \mathbf{F}(x, y)$. We will return to this idea in a few weeks, after we have developed the idea of *surface integrals* and *flux*.

Examples.

- Let $\mathbf{F} = \langle 1, 1 \rangle$; this is an example of a constant vector field, and can be thought of as corresponding to the flow of a fluid which moves at a constant rate. Then $\nabla \cdot \mathbf{F} = 0$. This corresponds to the idea that divergence should measure whether a point is a source or a sink; if the rate of flow of a fluid is constant, then the amount of fluid leaving a point is equal to the amount going in and so we should expect $\nabla \cdot \mathbf{F} = 0$.

- Let $\mathbf{F} = \langle x, 0\rangle$. This is a vector field which can be thought of as representing a fluid which only flows in the x-direction. A quick calculation shows that $\nabla \cdot \mathbf{F} = 1$. This represents the fact that, at each point, more fluid is flowing out of the point than into it; we can see this intuitively since if we sketch this vector field, the arrows going into a point are slightly smaller than the arrows going out of the point.
- Let $\mathbf{F} = \langle y, 0\rangle$. This vector field has $\nabla \cdot \mathbf{F} = 0$. Although it is non-constant, the divergence can still zero because in a sketch of the vector field, the arrows leaving a point are all the same length as the arrows entering the point.
- Let $\mathbf{F} = \langle -y, x\rangle$. Recall that this vector field looks like a fluid which rotates in a counterclockwise, circular direction. Again, a quick calculation shows that $\nabla \cdot \mathbf{F} = 0$, which represents the fact that the arrows leaving a point are of the same length as the arrows entering a point.

2. The curl

We now turn our attention away from the divergence and discuss another operator, known as the *curl*. Unlike the divergence, we can only define the curl of a vector field $\mathbf{F} = \langle P, Q, R\rangle$ which lives in $\mathbb{R}^3$. Given such a vector field, the curl of $\mathbf{F}$, written curl $\mathbf{F}$ or $\nabla \times \mathbf{F}$, is defined to be the vector field

$$\nabla \times \mathbf{F} = \left\langle \frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}, -\left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right), \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right\rangle.$$

An alternate way of remembering this definition is to interpret the notation $\nabla \times \mathbf{F}$ in the following way: if we think of ∇ as representing the 'vector'

$$\nabla = \langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\rangle,$$

and we take the 'cross product' of ∇ with $\mathbf{F} = \langle P, Q, R\rangle$, we end up with a 3×3 'determinant'

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial_x & \partial_y & \partial_z \\ P & Q & R \end{vmatrix}.$$

If one expands this determinant in the usual way, taking care to interpret a 'product' like $\partial_x Q$ as the partial derivative Q_x, one ends up with exactly the original definition of the curl.

This is obviously much more complicated to calculate than the divergence of a vector field, but it also turns out to represent some sort of property of a vector field which might be interesting to understand. Suppose we've calculated $\nabla \times \mathbf{F}$ at some point (x, y, z). If we place a paddle whose axis is oriented in the direction of $\nabla \times \mathbf{F}$, and pretend that $\mathbf{F}$ represents the flow of a fluid, then the paddle will rotate. The magnitude of $\nabla \times \mathbf{F}$ represents the rate of rotation of the paddle, while the direction is chosen in such a way so as to match the 'right-hand rule': if we curl the fingers of our right hand in the direction of rotation of a vector field, the thumb should point in the direction of $\nabla \times \mathbf{F}$.

Examples.

- Let $\mathbf{F} = \langle 1, 1, 1\rangle$ be a constant vector field; again this can be thought of as representing a fluid flowing at a constant rate. Then $\nabla \times \mathbf{F} = \mathbf{0}$. This accords with the idea that curl measures rotations, since nowhere do we see any rotational tendencies in this vector field.
- Let $\mathbf{F} = \langle -y, x, 0\rangle$. This is a three-dimensional version of the rotational vector field we looked at earlier, with all the rotation occurring in the xy plane (or planes parallel to this plane). First, we calculate $\nabla \times \mathbf{F}$:

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial_x & \partial_y & \partial_z \\ -y & x & 0 \end{vmatrix} = \langle 0, 0, \partial_x x - \partial_y(-y)\rangle = \langle 0, 0, 2\rangle$$

  This corresponds to the fact that this field rotates around the z-axis. $\nabla \times \mathbf{F}$ points in the positive z direction, which accords with the fact that if we curl our fingers in the right hand in the counterclockwise direction, the thumb points in the positive z direction.
- Let $\mathbf{F} = \langle y, 0, 0\rangle$. Then $\nabla \times \mathbf{F}$ is given by

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial_x & \partial_y & \partial_z \\ y & 0 & 0 \end{vmatrix} = \langle 0, 0, -1\rangle.$$

  Even though on the surface there looks like there is no rotation in $\mathbf{F}$, if we place a paddle with axis in the z direction, because there is more flow on the top paddle than the bottom paddle, rotation will still occur. A bit of thought will show that this paddle rotates in the clockwise direction, so the right-hand rule tells us that we should expect $\nabla \times \mathbf{F}$ to point in the negative z direction, which is what we calculated.

In any event, the above examples are approximations to the idea of curl as measuring the rotational tendency of a vector field. In a few weeks, we will learn a more precise formulation of this idea.

3. Properties of the divergence and curl

We've seen how to calculate the curl and divergence of a vector field, and also discussed some basic, intuitive interpretations of them. We now want to discuss some basic properties of the divergence and curl, some of which are generalizations of properties for vector fields in $\mathbb{R}^2$ that we are familiar with.

One interesting property is that the divergence of the curl of a vector field is always equal to 0. That is,

$$\nabla \cdot (\nabla \times \mathbf{F}) = \mathbf{0},$$

at least when $\mathbf{F}$ is a C^2 vector field (the components of $\mathbf{F}$ have continuous second-order partial derivatives). This can be shown by a direction calculation of $\nabla \cdot (\nabla \times \mathbf{F})$, where we use the fact that second-order mixed partial derivatives will be equal to each other.

Another property is that the curl of a gradient is always equal to 0. That is, if $f(x, y, z)$ is a C^2 scalar function on $\mathbb{R}^3$, then

$$\nabla \times (\nabla f) = \nabla \times \nabla f = \mathbf{0},$$

regardless of what f actually is. Again, one shows this by direct calculation, along with the use of the fact that second-order mixed partial derivatives of f will be equal.

Recall that given a vector field $\mathbf{F} = \langle P, Q \rangle$ in $\mathbb{R}^2$, a necessary condition for $\mathbf{F}$ to be conservative was that $P_y = Q_x$, and that this condition was sufficient if $P_y = Q_x$ and $\mathbf{F}$ was defined on a simply connected set D. The condition $\nabla \times \mathbf{F} = \mathbf{0}$ is the three-dimensional analogue of the condition $P_y = Q_x$. That is, a necessary condition for a vector field $\mathbf{F} = \langle P, Q, R \rangle$ on $\mathbb{R}^3$ to be conservative (that is, of the form ∇f), is $\nabla \times \mathbf{F} = \mathbf{0}$. Furthermore, this condition is sufficient if $\mathbf{F}$ is defined on a simply-connected set D and $\nabla \times \mathbf{F} = \mathbf{0}$ on D.

Example. The rotational vector field $\mathbf{F} = \langle -y, x, 0 \rangle$ is not conservative, since $\nabla \times \mathbf{F} = \langle 0, 0, 2 \rangle$, while the vector field $\mathbf{F} = \langle x, y, z \rangle$ is conservative, because $\nabla \times \mathbf{F} = \mathbf{0}$ and this is true for all (x, y, z) in $\mathbb{R}^3$, which is simply connected.

In both these cases, we could also have shown that $\mathbf{F}$ was conservative/not conservative by calculating various partial integrals, and either reconstructing a potential function or showing that no potential function could exist by reaching a contradiction. However, for more complicated vector fields, calculating partial integrals might be much harder than differentiation, if not impossible.

We also mentioned that $\nabla \cdot (\nabla \times \mathbf{F}) = 0$ for any (C^2) vector field $\mathbf{F}$. What if we are given a vector field $\mathbf{G}$ such that $\nabla \cdot \mathbf{G} = 0$? Can we conclude that there exists a vector field $\mathbf{F}$ such that $\mathbf{G} = \nabla \times \mathbf{F}$? Again, the answer is, not necessarily – $\nabla \cdot \mathbf{G} = 0$ is a necessary, but not sufficient, condition for there to exist some vector field $\mathbf{F}$ such that $\mathbf{G} = \nabla \times \mathbf{F}$. However, assuming a different topological condition than simple-connectedness, which we will not detail here, for the region D that $\mathbf{F}$ is defined on, in some situations one can conclude that $\mathbf{G} = \nabla \times \mathbf{F}$ if $\nabla \cdot \mathbf{G} = 0$. We will not say much more about this, other than that this is hinting at deep ideas in geometry. (See, for instance, this Wikipedia article about *de Rham cohomology*.)

We finally remark that the divergence of a gradient, such as

$$\nabla \cdot \nabla f(x, y) = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2},$$

is sometimes called the *Laplacian* of f. This is sometimes written as $\nabla^2 f$ or Δf. We will not talk about the Laplacian in this class, but it is of fundamental importance in mathematics, physics, and engineering. Its relevance in real-life stems from the fact that solutions to the equation

$$\Delta f = \lambda f,$$

where λ is some constant, are of great importance in understanding the behavior of heat flow and waves (both the fluid variety and vibrations of strings and membranes), among other natural phenomena. The study of solutions to this equation, and the use of these functions to understand more complicated functions, falls under a branch of mathematics called *Fourier analysis*, and might very well be the most important mathematical invention of the past two centuries. If you are interested in learning more, a class in partial differential equations is a good place to start.