LINE INTEGRALS OF VECTOR FUNCTIONS

Contents

1. The Fundamental Theorem of Calculus for line integrals

Suppose that $\mathbf{F}$ is a conservative vector field; that is, $\mathbf{F} = \nabla f$ for some scalar function f. We initially assume that this relation holds true for all (x, y) in $\mathbb{R}^2$, although later we will see how we can relax this condition. Then the following theorem holds true:

Theorem. (The Fundamental Theorem of Calculus for line integrals) Let $\mathbf{F} = \nabla f$ be a continuous vector field for some differentiable scalar function $f(x, y)$ on $\mathbb{R}^2$. Let C be a curve parameterized by $\mathbf{r}(t), a \leq t \leq b$. Then

$$\int_C \mathbf{F}\, d\mathbf{r} = f(\mathbf{r}(b)) - f(\mathbf{r}(a)).$$

Remarks.

- Notice the strong resemblance of this theorem to the usual FTC. Some sort of definite integral is equal to the value of a function at one point subtracted from the value of a function at another point. The role of an antiderivative for $\mathbf{F}$ is played by the scalar function f, and the role of the endpoints of the interval $[a, b]$ are played by the endpoints of C, $\mathbf{r}(a), \mathbf{r}(b)$.
- One consequence of this theorem is that if $\mathbf{F}$ is conservative on $\mathbb{R}^2$, then the value of a line integral of $\mathbf{F}$ along C does not actually depend on the path taken by C, but only on the endpoints of C. That is, a phenomenon like that which we observed in the previous set of examples can never take place for a conservative vector field. We say that a line integral of $\mathbf{F}$ over C is *independent of path* if its values only depend on the endpoints of C. In this terminology, the line integral of a conservative vector field is independent of path for all curves C.
- A special case of the above case occurs when we calculate the line integral of $\mathbf{F}$ along a curve C whose starting and endpoints are the same. If C is such a curve, we call C a closed curve, and the line integral of a conservative vector field along a closed curve C is equal to

$$\int_C \mathbf{F}\, d\mathbf{r} = f(\mathbf{r}(b)) - f(\mathbf{r}(a)) = 0,$$

since $\mathbf{r}(b) = \mathbf{r}(a)$. Therefore, the line integral of a conservative vector field along a closed path is always equal to 0, regardless of the shape of the closed path.
- The resemblance of this theorem to the usual FTC is not accidental. As a matter of fact, the proof of this theorem (which we do not cover here) uses the usual FTC along with the chain rule applied to $\mathbf{F}(x(t), y(t))$.

One nice application of the above theorem is that it becomes possible to calculate the line integral of conservative vector fields even when the path C is very strange or complicated.

Example. Let $\mathbf{F} = \langle ye^x, e^x \rangle$, and let C be given by parameterization $\mathbf{r}(t) = \langle t^{17}, \cos^3 \pi t \rangle, 0 \leq t \leq 1$. Calculate the line integral of $\mathbf{F}$ along C.

If you try to calculate this line integral directly, you will end up with a complete mess of an expression in the function t. The much faster way to calculate this line integral is to recognize $\mathbf{F}$ as the gradient of $f(x, y) = ye^x$. One way of finding this function is to pretend that there is an $f(x, y)$ with $\nabla f = \mathbf{F}$. If this is the case, then $f_x = ye^x, f_y = e^x$. In particular, the latter expression forces $f(x, y)$ to have the form $f(x, y) = ye^x + g(y)$, for some function $g(y)$ solely of y, while the former expression requires $f(x, y) = ye^x + h(x)$, for some function $h(x)$ solely of x. Since these two expressions must be equal, we have $f(x, y) = ye^x$, since $g(y) = h(x)$ is only possible if both expressions are constant. (We could have chosen $f(x, y) = ye^x + C$ for any constant C, but this does not affect the final answers.)

Since $\mathbf{F}$ is the gradient of a function differentiable on all of $\mathbb{R}^2$, we have

$$\int_C \mathbf{F}\, d\mathbf{r} = f(\mathbf{r}(1)) - f(\mathbf{r}(0)).$$

Since $\mathbf{r}(1) = \langle 1^{17}, \cos^3 \pi \rangle = \langle 1, -1 \rangle, \mathbf{r}(0) = \langle 0^{17}, \cos^3 0 \rangle = \langle 0, 1 \rangle$, this expression is equal to

$$f(1, -1) - f(0, 1) = -e^1 - 1 = -(e + 1).$$

This example suggests one natural, very important question: given a vector field $\mathbf{F}$, is there a quick way of determining whether it is conservative? In this example we were able to take 'partial integrals' to find a potential function $f(x, y)$ for $\mathbf{F}$. However, there may be situations where we cannot take partial integrals.

When we stated the Fundamental Theorem of Calculus for line integrals, we didn't pay much attention to the domain over which $\mathbf{F} = \nabla f$; we more or less tacitly assumed that it was all of $\mathbb{R}^2$. However, there are many situations where we would like a more flexible statement of the FTC.

Let D be a subset of $\mathbb{R}^2$. If $\mathbf{F}$ is a vector field defined on D and there exists a scalar function f such that $\mathbf{F} = \nabla f$ on D, then the FTC still holds true for any line integrals of $\mathbf{F}$ on curves C which are contained entirely in D.

Another problem that came up when trying to apply the FTC was determining a potential function f for a conservative vector field $\mathbf{F}$. Consider the following example:

Example. Let $\mathbf{F} = \langle 2x, 3y^2 \rangle$. Show that $\mathbf{F}$ is conservative and find a potential function for it. Again, the 'partial integration' technique shows that $f(x, y) = x^2 +$

$g(y) = y^3 + h(x)$. Notice that in this example, the only way this is true is if $g(y) = y^3 + C, h(x) = x^2 + C$, so a potential function is given by $f(x, y) = x^2 + y^3 + C$.

2. Properties of conservative vector fields

Let's quickly review the properties of conservative vector fields we've seen so far:

A vector field $\mathbf{F}$ is called a conservative vector field of $\mathbf{F}$ on a domain D if it is equal to the gradient of some scalar function f on D; that is, $\mathbf{F} = \nabla f$ for all points in D. A conservative vector field satisfies the fundamental theorem of calculus for line integrals, which says that if a path C, lying entirely in D, is parameterized by $\mathbf{r}(t), a \leq t \leq b$, then

$$\int_C \mathbf{F} \cdot d\mathbf{r} = f(\mathbf{r}(b)) - f(\mathbf{r}(a)).$$

This is analogous to the usual FTC, and can be used to calculate line integrals of conservative vector fields over complicated paths if one can calculate a potential function for that field.

A consequence of the FTC for line integrals is that if $\mathbf{F}$ is conservative on D, then $\mathbf{F}$ is *path independent* on D, which means that the value of a line integral of $\mathbf{F}$ along C only depends on the start and end point of C, and not on the path in between. More precisely, if two paths C_1, C_2 both lie entirely in D and start and end at the same point, then

$$\int_{C_1} \mathbf{F} \cdot d\mathbf{r} = \int_{C_2} \mathbf{F} \cdot d\mathbf{r}.$$

Remember that this might not happen for non-conservative vector fields! In the first examples we computed we gave a simple example of a non-conservative vector field whose line integrals along two different paths were different, even though the paths had the same start and end point.

A consequence of path independence is that the line integral of a conservative vector field $\mathbf{F}$ along any *closed path* in D is always equal to 0. A closed path is a path whose starting and end point are identical; ie, $\mathbf{r}(b) = \mathbf{r}(a)$. To see why this is so, suppose C is a closed path with some orientation. We can split this closed path into two separate paths C_1, C_2 by selecting any point, say $\mathbf{r}(c)$, (not equal to the start or end point) and splitting the path in half at that point. Let C_1 is the path with $a \leq t \leq c$, and C_2 the path with $c \leq t \leq b$. Although the starting point of C_1 is the end point of C_2, we can change this by reversing the orientation on C_2. Call this curve $-C_2$ (it traces out the exact same curve as C_2, but has the opposite orientation of C_2). Then $C_1, -C_2$ are two paths, lying entirely in D, which start and end at the same point. Since $\mathbf{F}$ is conservative on D, we have

$$\int_{C_1} \mathbf{F} \cdot d\mathbf{r} = \int_{-C_2} \mathbf{F} \cdot d\mathbf{r} = -\int_{C_2} \mathbf{F} \cdot d\mathbf{r},$$

where we use the fact that reversing the orientation of a path flips the sign of a line integral. In particular, this tells us that

$$\int_{C_1} \mathbf{F} \cdot d\mathbf{r} + \int_{C_2} \mathbf{F} \cdot d\mathbf{r} = \int_C \mathbf{F} \cdot d\mathbf{r} = 0.$$

As a matter of fact, all the arguments above could be reversed to show that if $\mathbf{F}$ is a vector field for which $\int_C \mathbf{F} \cdot d\mathbf{r} = 0$ for every closed path C in some domain D, then $\mathbf{F}$ will be path-independent in D.

In summary, the FTC tells us that a conservative vector field on D will also be path-independent. Suppose that D is an *open, connected* set. Intuitively, a set is open if it does not have any boundary points; more precisely, an open set is one in which for every point $x \in D$, there is a disc containing x which also lies entirely in D. Sets with boundary points cannot satisfy this property since any disc containing a boundary point will also contain points outside of D. A set D is *connected* if, given any two points in a connected set, there exists a path contained entirely in D joining those two points. Intuitively, a connected set consists only of 'one piece', and not more pieces.

Examples.

- The open disc $x^2 + y^2 < 1$ is open, because it has no boundary points, while the (closed) disc $x^2 + y^2 \leq 1$ is not open, since it has boundary points.
- Both the above sets are connected, but the set consisting of points which satisfy either $x^2 + y^2 \leq 1$ or $x + y \geq 10$ is not connected since it consists of two separate parts. If you pick one point in the disc portion of the set and another point in the $x + y \geq 10$ part of the set, there is no path which stays in the set which joins the two points together.
- In general, the properties of being open and connected are not correlated in any way with each other; that is, knowing that a set is open tells you nothing about whether the set is connected and vice versa.

In practice, for most sets you see it will be easy to determine whether the set is open, connected, or both. Suppose D is an open, connected set. Then it turns out that the property of a vector field $\mathbf{F}$ being conservative on D is actually equivalent to path-independence on D:

Theorem. Suppose D is an open connected set, and that $\mathbf{F}$ is path-independent on D. Let (a, b) be any point in D. Then $\mathbf{F}$ is conservative on D, with potential function $f(x, y)$ defined by

$$f(x, y) = \int_C \mathbf{F} \cdot d\mathbf{r},$$

where C is any path contained in D which starts at (a, b) and ends at (x, y).

We will not prove this theorem, but notice that the connectedness property as well as the path-independence property are both required in order for the definition of the potential function $f(x, y)$ to make any sense. The fact that D is open is used to check that $\nabla f = \mathbf{F}$; see the textbook for more details.

We would like to be able to determine whether $\mathbf{F}$ is conservative without too much difficulty. However, the path-independence property for conservative fields does not help at all with this problem, since in practice it is impossible to check that an integral is independent of path for EVERY choice of starting and end point and EVERY choice of path connecting these two points. In one example we saw how we could try to calculate 'partial integrals' to either find a potential function, or rule out its existence. However, this requires calculating integrals, which in general can be a fairly difficult problem.

In an earlier example, we showed that a field was not conservative by assuming that it was, and then showing that this led to a contradiction. More specifically, suppose $\mathbf{F} = \langle P, Q \rangle$ is conservative, so that $\mathbf{F} = \nabla f$, and make the additional assumption that $\mathbf{F}$ is C^1; ie, P, Q have continuous first-order partial derivatives. Then $f_x = P, f_y = Q$, and we can apply Clairaut's Theorem to conclude that $f_{xy} = P_y = f_{yx} = Q_x$. In other words, if $\mathbf{F} = \langle P, Q \rangle$ is conservative and C^1, then

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}.$$

Any conservative vector field satisfies the above property, which only involves taking derivatives, not integrals. As such, this looks like it is a better test for whether a vector field is conservative or not than anything else we know. However, there is one major problem: not every field which passes this test is conservative! In other words, if $P_y \neq Q_x$ for even one point in D, then we know that $\mathbf{F}$ is not conservative on D, but even if $P_y = Q_x$ everywhere on D, we cannot necessarily conclude that $\mathbf{F}$ is conservative.