TRIPLE INTEGRATION

Contents

1. Interchanging order of integration

Just like in the two-dimensional case, it is possible to interchange the order of integration when calculating triple integrals – as a matter of fact, it is sometimes very helpful or even necessary to do so. We look at one example where we can interchange order of integration, and then use this example again when discussing cylindrical coordinates.

Example. Consider the region E satisfying the inequalities $\sqrt{x^2+y^2} \leq z \leq 1$. Write down iterated integrals which equal the triple integral of a function $f(x,y,z)$ over the region E with order $dz\,dy\,dx$ and $dx\,dy\,dz$.

We begin by drawing a sketch of E. The surface $z = \sqrt{x^2+y^2}$ is a cone with vertex at the origin, which gets wider as z increases. Therefore, the region given by $\sqrt{x^2+y^2} \leq z \leq 1$ is an inverted cone whose vertex lies at the origin, and whose upper boundary is given by a disc $x^2+y^2 \leq 1, z = 1$.

If we want to write an iterated integral with order of integration $dz\,dy\,dx$, we start by determining bounds on z in terms of x, y. These are basically given to us in this problem, as $\sqrt{x^2+y^2} \leq z \leq 1$. To determine the bounds on y, x, we examine the projection of E onto the xy plane. In this situation, we see that the projection of E is the disc $x^2+y^2 \leq 1$. Therefore, y satisfies the inequalities $-\sqrt{1-x^2} \leq y \leq \sqrt{1-x^2}$, while x satisfies the inequalities $-1 \leq x \leq 1$. Of course, the intersection of E with the xy plane is just the origin, so in this example the projection of E onto the xy plane is something larger than the intersection of E with the xy plane. In any case, the iterated integral in question is

$$\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\int_{\sqrt{x^2+y^2}}^{1} f(x,y,z)dz\,dy\,dx.$$

If we want an iterated integral with order of integration $dx\,dy\,dz$, we first start by determining what inequality x must satisfy as a function of y, z. The x coordinates of points in E are evidently bounded by the boundary of the cone, so will be bounded by the functions of x we obtain when we solve for x in $x^2+y^2 = z^2$. This yields $x = \pm\sqrt{z^2-y^2}$, so $-\sqrt{z^2-y^2} \leq x \leq \sqrt{z^2-y^2}$. To determine bounds for y, z, we begin by projecting E onto the yz plane. In this case, one can see that the projection will be a triangle whose vertices have coordinates $(0,0), (-1,1), (1,1)$.

The inequalities which y satisfies are thus given by $-z \leq y \leq z$. Finally, $0 \leq z \leq 1$, so the iterated integral in question is

$$\int_0^1 \int_{-z}^{z} \int_{-\sqrt{z^2-y^2}}^{\sqrt{z^2-y^2}} f(x,y,z)\,dx\,dy\,dz.$$

Therefore, we see that writing down an iterated integral equal to a triple integral can be a fairly lengthy and difficult process (and we haven't even done any integrating yet!). We first need to determine the inequalities that the first variable we will integrate satisfy, in terms of the other two remaining variables. This often involves identifying the surfaces which bound the region E, and then solving for the variable in question in terms of the other two variables.

We then need to determine the projection of the region E onto the plane spanned by the remaining two variables. In general this is a very difficult question, and is only solvable in very special situations. You certainly should not expect to be able to solve this question for 'arbitrary' regions E, so if you are asked to write down an iterated integral of a region E then there must be a way to somewhat easily determine the projection of E onto one of the three coordinate planes. Once this projection is determined, writing down the bounds for the remaining two variables reduces to a two-dimensional question which we have encountered many times when evaluating double integrals.

2. Cylindrical coordinates

We will now take a look at evaluating triple integrals using coordinate systems different from rectangular coordinates. This is exactly the analogue of when we studied polar coordinates with double integrals. We begin with *cylindrical coordinates*, which are very closely related to polar coordinates.

A point (x,y,z), in rectangular coordinates, has cylindrical coordinates (r,θ,z), if $x = r\cos\theta, y = r\sin\theta$. That is, we extend polar coordinates to three dimensions by simply tacking on an additional z coordinate. The reason why these coordinates are called cylindrical coordinates is clear if we look at the surfaces determined by equations of the form $r = C$, for various constants C. These are cylinders which are centered around the z-axis.

We want a formula which relates an integral over rectangular coordinates to a corresponding integral over cylindrical coordinates. If a region E in (x,y,z) space is given by cylindrical inequalities $0 \leq a \leq r \leq b, \alpha \leq \theta \leq \beta(0 \leq \beta - \alpha \leq 2\pi), z_1 \leq z \leq z_2$, then we have an inequality of integrals

$$\iiint\limits_E f(x,y,z)\,dV = \int_\alpha^\beta \int_a^b \int_{z_1}^{z_2} f(r\cos\theta, r\sin\theta, z) r\,dz\,dr\,d\theta.$$

Notice the similarity of this formula to the formula relating integration over polar coordinates to rectangular coordinates in two dimensions. There is the same factor of r appearing in the integrand, and we replace each of x, y, z in the integrand with $r\cos\theta, r\sin\theta, z$, respectively.

Of course, a similar formula holds true in the more general case where z might be a function of r, θ, and r a function of θ.

Cylindrical coordinates are really useful in problems involving not only cylinders, but any situations where expressions of the form $x^2 + y^2$ appear.

Example. Consider the cone described in the last problem – the set of points described by the inequalities $\sqrt{x^2+y^2} \leq z \leq 1$. Write down a triple integral in cylindrical coordinates which equals the volume of this cone.

Notice that the inequalities on r, z are given by $0 \leq r \leq z \leq 1$, since $x^2 + y^2 = r^2$. In particular, the inequalities for z are $r \leq z \leq 1$. The inequalities on θ are given by $0 \leq \theta \leq 2\pi$, since any cross-section of the cone by a plane $z = C$ is a disc. From the first set of inequalities we see that $0 \leq r \leq 1$.

$$\iiint_E dV = \int_0^{2\pi} \int_0^1 \int_1^r r\, dz\, dr\, d\theta.$$

Notice that this integral is much easier to calculate than the various iterated integrals we found in the previous example:

$$\int_0^{2\pi} \int_0^1 \int_r^1 r\, dz\, dr\, d\theta = \int_0^{2\pi} \int_0^1 r(1-r)\, dr\, d\theta = \int_0^{2\pi} \left(\frac{r^2}{2} - \frac{r^3}{3} \Big|_{r=0}^{r=1} \right) d\theta = \int_0^{2\pi} \frac{1}{6} = \frac{\pi}{3}.$$

Suppose that this cone is a solid with density given by $\rho(x, y, z) = z$, and we want to calculate the mass of the solid. Then we want to evaluate

$$\iiint_E z\, dV = \int_0^{2\pi} \int_0^1 \int_r^1 rz\, dz\, dr\, d\theta.$$

Again, this integral is easy to evaluate (contrast it to the corresponding integral you would need to evaluate in rectangular coordinates):

$$\int_0^{2\pi} \int_0^1 \int_r^1 rz\, dz\, dr\, d\theta = \int_0^{2\pi} \int_0^1 r \frac{z^2}{2} \Big|_{z=r}^{z=1} dr\, d\theta = \int_0^{2\pi} \int_0^1 r\left(\frac{1}{2} - \frac{r^2}{2}\right) dr\, d\theta = \int_0^{2\pi} \frac{r^2}{4} - \frac{r^4}{8} \Big|_{r=0}^{r=1} d\theta = \frac{\pi}{4}.$$

Example. Consider the solid E bounded by the surfaces $z = \sqrt{x^2+y^2}$, $z = 2 - x^2 - y^2$. Write down a triple integral equal to the volume of this solid, and evaluate it.

First, we begin by sketching these two surfaces. The first surface is a cone (as we have seen), while the second surface is a elliptic (even circular, in this case) paraboloid which is up-side down. Furthermore, the cone forms the bottom boundary of the surface while the paraboloid forms the top. Evidently, this solid looks somewhat like an ice-cream cone.

We want to determine the projection of E onto the xy plane. To do this, we should perhaps start by finding the intersection of the two surfaces. We set both functions of x, y equal to each other and obtain $\sqrt{x^2+y^2} = 2 - x^2 - y^2$. If we let $r = \sqrt{x^2+y^2}$ (which is natural as we are going to eventually end up using cylindrical coordinates anyway), this equation becomes $r = 2 - r^2$. This is a quadratic for r, and the solutions to this equation are $r = 1, -2$. We discard the $r = -2$ solution since $r \geq 0$, and so the two surfaces intersect when $r = 1$. Furthermore, when $r = 1, z = r = 2 - r^2 = 1$, so the intersection of these two surfaces is a circle given by the equations $x^2 + y^2 = 1, z = 1$.

Amongst all points in E, r is maximal at the boundary of this circle, so the projection of E onto the xy plane is the disc $x^2 + y^2 \leq 1$. In cylindrical coordinates, this corresponds to inequalities $0 \leq \theta \leq 2\pi, 0 \leq r \leq 1$. Therefore, the triple integral which equals the volume of this solid is

$$\iiint\limits_E dV = \int_0^{2\pi} \int_0^1 \int_r^{2-r^2} r\, dz\, dr\, d\theta.$$

We evaluate this integral:

$$\int_0^{2\pi} \int_0^1 \int_r^{2-r^2} r\, dz\, dr\, d\theta = \int_0^{2\pi} \int_0^1 r(2-r^2-r)\, dr\, d\theta = 2\pi \left(r^2 - \frac{r^3}{3} - \frac{r^4}{4} \Big|_{r=0}^{r=1} \right) = 2\pi \left(1 - \frac{1}{3} - \frac{1}{4} \right) = \frac{5\pi}{6}$$

Of course, we could have instead just calculated a double integral over the disc $x^2 + y^2 \leq 1$, using polar coordinates, to find this volume. However, setting up a triple integral has the advantage of letting us also calculate quantities like mass, moments, moment of inertia, etc., of a solid in the shape of E – we would not need to adjust the bounds of integration, only the integrand.