A GUIDE TO UNDERSTANDING MATHEMATICIANS

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"...some of you may have had occasion to run into mathematicians and to wonder, therefore, how they got that way..." –Tom Lehrer

Mathematicians don't always use words to mean exactly the same thing that they mean in everyday speech. Here is a quick guide to the grammar of mathematics. When I say that something is a *statement,* I mean that it is a sentence whose truth is unambiguous like "2 is a prime number" or "there is no positive number less than $\frac{1}{2}$."

1. Boolean connectives

"Not": This is straightforward: If P is a statement, then "not P" is true when P is false and false when P is true. You may see this symbolized by $\neg$.

"And": This one also corresponds to typical English usage: If P and Q are statements, then "P and Q" is true precisely when P is true and Q is true. If either of them is false, then "P and Q" is false as well. A mathematician might write $P \wedge Q$ or $P \,\&\, Q$ to mean "P and Q."

"Or": This is the first one where standard English usage differs from mathematical usage. If a mathematician says "P or Q," he or she means that *at least one* of P and Q must be true. Normally, "P or Q" means that one of P and Q is true, but not both. (For instance, to a mathematician, "Would you like cake or ice cream?" could legitimately be answered with "Yes, both." Most people would say that sounded greedy.) The only way "P or Q" can be false for a mathematician is if P and Q are both false. You would write "P or Q" as $P \vee Q$.

"If...then...": This doesn't mean quite the same thing to a mathematician that it means to a nonmathematician, either. The only way that "If P, then Q" can be false is if P is true and Q is false. Simply put, you can't conclude something false if you start with something true. On the other hand, if you start with something false, you can conclude anything you like, so as long as P is false, "If P, then Q" is true. This is normally written as $P \to Q$, and there are several different ways this can be said in English: "P implies Q" and "Q if P" are the other two common ones, but you might also see "P only if Q," "P is a sufficient condition for Q," or "Q is a necessary condition for P."

There are two very important points about $\to$:

- $P \to Q$ is not the same as $Q \to P$ (the *converse* of $P \to Q$). To see this, think about the sentences "If n is divisible by 6, then n is divisible by 3" and "If n is divisible by 3, then n is divisible by 6." The first is true and the second is false.

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- $P \to Q$ is equivalent to $\neg Q \to \neg P$. The latter is called the *contrapositive*. Sometimes, if we want to show $P \to Q$, we will show $\neg Q \to \neg P$ instead because it's easier.

"If and only if": This one is easy: "P if and only if Q" is true when $P \to Q$ and $Q \to P$. It's often abbreviated "iff" and is written as $P \leftrightarrow Q$.

2. Quantifiers

"For all": This one is not very mysterious. "For all x, $x^2 \geq 0$" means that every x has a nonnegative square, and it would be written mathematically as $(\forall x)x^2 \geq 0$ ("for all" is written as a $\forall$). This really should be made more explicit. If I am only considering real x, this is true, but if I am considering complex x, this is false. It would be better to give a "domain" explicitly and write "For all $x \in \mathbb{R}$, $x^2 \geq 0$." However, often the "domain" of $\forall$ will be unspecified, and you will have to rely on context: if $f : \mathbb{R}^2 \to \mathbb{R}$, then $(\forall x, y)f(x, y) < 3$ really means $(\forall x, y \in \mathbb{R})f(x, y) < 3$.

"There exists": This means exactly what it says and no more. "There exists x such that $x^2 \geq 0$" means that there is some x with a nonnegative square. (The same "domain" issues apply here: we would probably read this as "there is a real number x with a nonnegative square.) While it's true that *every* real number x has a nonnegative square, we may wish to say less than is actually true sometimes. Similarly, "there exists" doesn't mean "there is *only one*": I can say "There exists a book in the Dartmouth library" without suggesting that there is only one. "There exists" is symbolized by $\exists$, so the statement above could be written as $(\exists x)x^2 \geq 0$.

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