

Vector Fields

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Vector Fields

Definition: A **vector field** on \mathbb{R}^n is a mapping

$$\mathbf{F} : X \subseteq \mathbb{R}^n \rightarrow \mathbb{R}^n.$$

Flow Lines

A **flow line** of a vector field \mathbf{F} is a differentiable path \mathbf{x} such that

$$\mathbf{x}'(t) = \mathbf{F}(\mathbf{x}(t))$$

Most important vector field: Gradient field

The most important example of a vector field is the gradient of a scalar valued function, $f : X \subseteq \mathbb{R}^n \rightarrow \mathbb{R}$

$$\nabla f(x_1, \dots, x_n) = \left(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \dots, \frac{\partial f}{\partial x_n} \right).$$

f is called **potential function**.

Question: Given a vector field \mathbf{F} is it possible to find an f such that

$$F = \nabla f?$$

Conservative Vector Field

Let \mathbf{F} be a vector field. Then \mathbf{F} is called **conservative** if there is a differentiable function f such that

$$\nabla f = \mathbf{F}$$

f is called the **potential function** for \mathbf{F} .

The Del Operator

$$\nabla = \mathbf{i} \frac{\partial}{\partial x} + \mathbf{j} \frac{\partial}{\partial y} + \mathbf{k} \frac{\partial}{\partial z}.$$

In general

$$\nabla = \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \dots, \frac{\partial}{\partial x_n} \right).$$

The Divergence

Let $\mathbf{F} : X \subseteq \mathbb{R}^n \rightarrow \mathbb{R}^n$ then the **divergence** is

$$\operatorname{div} \mathbf{F} = \nabla \cdot \mathbf{F} = \frac{\partial F_1}{\partial x_1} + \frac{\partial F_2}{\partial x_2} + \cdots + \frac{\partial F_n}{\partial x_n}$$

where F_i 's are the component functions of the vector field \mathbf{F} .

The Curl

Let $\mathbf{F} : X \subseteq \mathbb{R}^3 \rightarrow \mathbb{R}^3$ then the **curl** of \mathbf{F} is

$$\operatorname{curl} \mathbf{F} = \nabla \times \mathbf{F} = \det \begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{pmatrix}.$$

Incompressible vector fields

A vector field \mathbf{F} is called **incompressible** if $\operatorname{div} \mathbf{F} = 0$.

If $\mathbf{F} : X \subset \mathbb{R}^3 \rightarrow \mathbb{R}^3$ is a differentiable vector field. Then

$$\operatorname{div} (\operatorname{curl} \mathbf{F}) = 0.$$

This says that $\operatorname{curl} \mathbf{F}$ is an incompressible vector field.

Irrotational vector fields

A vector field \mathbf{F} in \mathbb{R}^3 is called **irrotational** if $\text{curl } \mathbf{F} = \mathbf{0}$.

If $f : X \subseteq \mathbb{R}^3 \rightarrow \mathbb{R}$ is differentiable, then

$$\text{curl } (\nabla f) = \mathbf{0}.$$

This says that the gradient of f is irrotational.