Review of vectors

January 5, 2009

Vectors in $\mathbb{R}^n$

In this class a **scalar** is simply a real number. An element in $\mathbb{R}$.

A **vector** in $\mathbb{R}^2$ is an ordered pair (x, y) of real numbers.

A **vector** in $\mathbb{R}^3$ is an ordered triple (x, y, z) of real number.

A **vector** in $\mathbb{R}^n$ is an ordered n-tuple $(x_1, x_2, \ldots, x_n)$ of n real numbers.

Operations on vectors

Vector Addition: Let $\mathbf{a} = (a_1, a_2, \ldots, a_n)$ and $\mathbf{b} = (b_1, b_2, \ldots, b_n)$ in $\mathbb{R}^n$ then their **sum** is

$$\mathbf{a} + \mathbf{b} = (a_1 + b_1, a_2 + b_2, \ldots, a_n + b_n)$$

Scalar Multiplication: Let $\mathbf{a} = (a_1, a_2, \ldots, a_n)$ be a vector in $\mathbb{R}^n$ and k any scalar then

$$k\mathbf{a} = (ka_1, ka_2, \ldots, ka_n)$$

The standard basis vectors

The standard basis vectors in $\mathbb{R}^2$ are
$\mathbf{i} = (1, 0)$ and $\mathbf{j} = (0, 1)$.

The standard basis vectors in $\mathbb{R}^3$ are
$\mathbf{i} = (1, 0, 0)$ and $\mathbf{j} = (0, 1, 0)$ and $\mathbf{k} = (0, 0, 1)$.

The standard basis vectors in $\mathbb{R}^n$ are
$\mathbf{e}_1 = (1, 0, \ldots, 0)$, $\mathbf{e}_2 = (0, 1, 0, \ldots, 0), \ldots,$
$\mathbf{e}_n = (0, \ldots, 0, 1)$.

Vector equation for a line in $\mathbb{R}^3$

The vector parametric equation for a line through the point $P(b_1, b_2, b_3)$, with position vector $\vec{OP} = \mathbf{b} = (b_1, b_2, b_3)$, and parallel to $\mathbf{a} = (a_1, a_2, a_3)$ is

$$\mathbf{r}(t) = \mathbf{b} + t\mathbf{a}$$

The Dot Product

Let $\mathbf{a} = (a_1, \ldots, a_n)$ and $\mathbf{b} = (b_1, \ldots, b_n)$ be two vectors in $\mathbb{R}^n$. The **dot product** of $\mathbf{a}$ and $\mathbf{b}$ is

$$\mathbf{a} \cdot \mathbf{b} = a_1 b_1 + a2b_2 + \ldots + a_n b_n$$

When $n = 3$, $\mathbf{a} = (a_1, a_2, a_3)$ and $\mathbf{b} = (b_1, b_2, b_3)$ and $\mathbf{a} \cdot \mathbf{b} = a_1 b_1 + a_2 b_2 + c_1 c_2$.

Length, Angle and Projection

The **length** of a vector is $\|\mathbf{a}\| = \sqrt{\mathbf{a} \cdot \mathbf{a}}$

The **angle** between two vectors $\mathbf{a}$ and $\mathbf{b}$ is

$$\theta = cos^{-1}\left(\frac{\mathbf{a} \cdot \mathbf{b}}{\|\mathbf{a}\|\,\|\mathbf{b}\|}\right)$$

The **projection of vector $\mathbf{b}$ onto $\mathbf{a}$** is

$$\text{proj}_{\mathbf{a}}\mathbf{b} = \left(\frac{\mathbf{a} \cdot \mathbf{b}}{\mathbf{a} \cdot \mathbf{a}}\right)\mathbf{a}$$

$\frac{\mathbf{a} \cdot \mathbf{b}}{\|\mathbf{a}\|}$ is called the **scalar projection**.

The Cross Product for vectors in $\mathbb{R}^3$

For two vectors $\mathbf{a}$ and $\mathbf{b}$ in $\mathbb{R}^3$, the **cross product** of $\mathbf{a}$ and $\mathbf{b}$ is the vector $\mathbf{a} \times \mathbf{b}$ such that:

• The length is $\|\mathbf{a} \times \mathbf{b}\| = \|\mathbf{a}\|\|\mathbf{b}\| sin\theta$.

• The direction is determined by extending the fingers of your right hand along the vector $\mathbf{a}$ and curling them towards the vector $\mathbf{b}$, the thumb points in the direction of $\mathbf{a} \times \mathbf{b}$

Note: **If** $\mathbf{a}$ is parallel to $\mathbf{b}$, then $\mathbf{a} \times \mathbf{b} = \mathbf{0}$.

Determinants

Recall that a matrix is an array of numbers (in our case of real numbers).

The **determinant of a 2×2** matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is $\det(A) = |A| = ad - bc$.

The **determinant of a 3×3** matrix $A = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$ is
$\det(A) = |A| = aei + bfg + cdh - ceg - afh - bdi$.

Equation of a plane

A plane in $\mathbb{R}^3$ is determined by a point in the plane $P_0(x_0, y_0, z_0)$ and a vector $\mathbf{n} = (A, B, C)$ that is perpendicular (normal) to the plane.

$$\begin{aligned}\mathbf{n} \cdot \vec{P_0P} &= (A, B, C) \cdot (x - x_0, y - y_0, z - z_0) \\ &= A(x - x_0) + B(y - y_0) + C(z - z_0) = 0\end{aligned}$$