Functions in several variables and limits

January 9, 2006

Functions

Any function has three features:

- A **domain** set X;
- A **codomain** set Y;
- A **rule of assignment** - a rule that assign to each element x in X of the domain a "unique" element $f(x)$ in Y (the codomain).

Scalar-valued functions

Scalar valued functions are functions such that the domain is $X \subseteq \mathbb{R}^n$ and the codomain is $\mathbb{R}$ or a subset of $\mathbb{R}$.

REMARK: Review the definitions of range, one-to-one and onto.

The Graph of a function

Let $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ be a scalar valued function. Let $\mathbf{x} = (x_1, x_2, \ldots, x_n)$, then the **graph** of f is:

$$\text{Graph}f = \{(\mathbf{x}, f(\mathbf{x})) \,|\, \mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n\}$$

For example if $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$, then the graph of f is the set of points in $\mathbb{R}^3$ that look like $(x, y, f(x, y))$, where (x, y) is in $\mathbb{R}^2$.

Level Curves

Let f be a function of two variables and let c be a constant. The set of all (x, y) in the plane such that $f(x, y) = c$ is called a **level curve** of f with value c.

Definition of the limit

Definition: (Intuitive) Let $f : \mathbb{R}^n \to \mathbb{R}^m$, then

$$\lim_{\mathbf{x}\to\mathbf{a}} \mathbf{f}(\mathbf{x}) = \mathbf{L}$$

means that we can make $\|f(\mathbf{x}) - \mathbf{L}\|$ arbitrarily small (close to zero) by keeping $\|\mathbf{x} - \mathbf{a}\|$ sufficiently small (but not zero).

Rigorous definition of limit

Definition: Let $f : X \subseteq \mathbb{R}^n \to \mathbb{R}^m$ be a function. Then

$$\lim_{\mathbf{x} \to \mathbf{a}} \mathbf{f}(\mathbf{x}) = \mathbf{L}$$

means that given $\epsilon > 0$, you can find a $\delta > 0$ (often depending on ϵ) such that if $\mathbf{x} \in X$ and $0 < \|\mathbf{x} - \mathbf{a}\| < \delta$, then $0 < \|f(\mathbf{x}) - \mathbf{L}\| < \epsilon$

Properties of limits

1. If $\lim_{\mathbf{x}\to\mathbf{a}} \mathbf{f}(\mathbf{x}) = \mathbf{L}$ and $\lim_{\mathbf{x}\to\mathbf{a}} \mathbf{g}(\mathbf{x}) = \mathbf{M}$ then $\lim_{\mathbf{x}\to\mathbf{a}} (\mathbf{f} + \mathbf{g})(\mathbf{x}) = \mathbf{L} + \mathbf{M}$

2. If $\lim_{\mathbf{x}\to\mathbf{a}} \mathbf{f}(\mathbf{x}) = \mathbf{L}$, then $\lim_{\mathbf{x}\to\mathbf{a}} k\mathbf{f}(\mathbf{x}) = k\mathbf{L}$, where k is a scalar.

3. if $\lim_{\mathbf{x}\to\mathbf{a}} \mathbf{f}(\mathbf{x}) = \mathbf{L}$ and $\lim_{\mathbf{x}\to\mathbf{a}} \mathbf{g}(\mathbf{x}) = \mathbf{M}$ then $\lim_{\mathbf{x}\to\mathbf{a}} (\mathbf{fg})(\mathbf{x}) = \mathbf{LM}$

4. If $\lim_{\mathbf{x}\to\mathbf{a}} \mathbf{f}(\mathbf{x}) = \mathbf{L}$ and $g(\mathbf{x}) \neq 0$ for $\mathbf{x} \in X$, and $\lim_{\mathbf{x}\to\mathbf{a}} \mathbf{g}(\mathbf{x}) = \mathbf{M} \neq 0$, then $\lim_{\mathbf{x}\to\mathbf{a}} (\mathbf{f}/\mathbf{g})(\mathbf{x}) = \mathbf{L}/\mathbf{M}$.

Continuous Functions

Definition: Let $\mathbf{f} : X \subseteq \mathbb{R}^n \to \mathbb{R}^m$ and let $\mathbf{a} \in X$. Then, f is **continuous at a** if

$$\lim_{\mathbf{x} \to \mathbf{a}} \mathbf{f}(\mathbf{x}) = \mathbf{f}(\mathbf{a}).$$

$\mathbf{f}$ is called **continuous** if it is continuous at every point of the domain X.

- The sum $f + g$ of two continuous functions is a continuous function.

- The scalar multiple of a continuous function kf is continous.

- The product fg and the quotient f/g (when defined) of two continuous functions is continuous.