Surface Integrals

February 27, 2006

Standard Normal Vector

Given $\mathbf{X}(s,t) = (x(s,t), y(s,t), z(s,t))$, we have two tangent vectors:

$$\mathbf{T}_s = \frac{\partial \mathbf{T}}{\partial s} = (\frac{\partial x}{\partial s}, \frac{\partial y}{\partial s}, \frac{\partial z}{\partial s})$$

$$\mathbf{T}_t = \frac{\partial \mathbf{T}}{\partial t} = (\frac{\partial x}{\partial t}, \frac{\partial y}{\partial t}, \frac{\partial z}{\partial t})$$

Then the **standard normal vector** is

$$\boxed{\mathbf{N} = \mathbf{T}_s \times \mathbf{T}_t}$$

Smooth Surfaces

A parametrized surface S is **smooth** at a point $\mathbf{X}(s_0, y_0)$ if $\mathbf{X}$ is C^1 in a neighborhood of (s_0, t_0) and if

$$\mathbf{N}(s_0, y_0) = \mathbf{T}_s \times \mathbf{T}_t \neq \mathbf{0}$$

Surface Area

$$\boxed{\text{Surface area of } S = \iint_D \|\mathbf{T}_s \times \mathbf{T}_t\| \, ds dt.}$$

Scalar Surface Integral

The **scalar surface integral** of a continuous function f along a smooth parametrized surface $\mathbf{X}(s,t)$ over a bounded region D is

$$\begin{aligned} \iint_{\mathbf{X}} f\, dS &= \iint_D f(\mathbf{X}(s,t)) \|\mathbf{T}_s \times \mathbf{T}_t\|\, ds dt \\ &= \iint_D f(\mathbf{X}(s,t)) \|\mathbf{N}(\mathbf{s},\mathbf{t})\|\, ds dt \end{aligned}$$

Vector Surface Integral

The **vector surface integral** of a continuous vector field $\mathbf{F}(x, y, z)$ along a smooth parametrized surface $\mathbf{X}(s, t)$ over a bounded region D is

$$\boxed{\iint_{\mathbf{X}} \mathbf{F} \cdot d\mathbf{S} = \iint_{D} \mathbf{F}(\mathbf{X}(s, t)) \cdot \mathbf{N}(s, t)\, ds dt}$$

where $\mathbf{N}(s, t) = \mathbf{T}_s \times \mathbf{T}_t$.

Orientation of a surface

A **smooth orientable surface** is a surface S that has a tangent plane at every point (x, y, z) on S (except boundary points) and at each point there are two normal vectors $\mathbf{N}$ and $-\mathbf{N}$. In other words the surface has two sides. The choice of $\mathbf{N}$ gives S an orientation.

REMARK: The Möbious strip is an example of an **nonorientable** surface, it has only one side. We can only define the surface integrals for orientable surfaces!

Scalar Surface Integrals are independent of parametrization

Theorem: Let $\mathbf{X} : D_1 \to \mathbf{R}^3$ be a smooth parametrized surface and f any continuous function with domain containing $\mathbf{X}(D_1)$. If $\mathbf{Y}$ is a smooth reparametrization of X, then

$$\iint_{\mathbf{Y}} f \, dS = \iint_{\mathbf{X}} f \, dS.$$

Vector Surface Integrals and Reparametrizations

Let Y be a reparametrization of the smooth orientable surface X. Then

- If $\mathbf{Y}$ is orientation preserving, then
$$\iint_{\mathbf{Y}} \mathbf{F} \cdot d\mathbf{S} = \iint_{\mathbf{X}} \mathbf{F} \cdot d\mathbf{S}$$

- If $\mathbf{Y}$ is orientation reversing, then
$$\iint_{\mathbf{Y}} \mathbf{F} \cdot d\mathbf{S} = -\iint_{\mathbf{X}} \mathbf{F} \cdot d\mathbf{S}$$