Conservative Vector Fields

February 21, 2006

Conservative Vector Field

$\mathbf{F}$ is a **conservative vector field** if there is a scalar function f such that

$$\mathbf{F} = \nabla f$$

The function f is called a **potential function** for the vector field.

Definitions

- A path C is **simple** if it doesnt cross itself.

- A region D is **open** if it doesnt contain any of its boundary points.

- A region D is **connected** if we can connect any two points in the region with a path that lies completely in D.

Path-Independent Line Integrals

A continuous vector field $\mathbf{F}$ has **path-independent line integrals** if

$$\int_{C_1} \mathbf{F} \cdot d\mathbf{s} = \int_{C_2} \mathbf{F} \cdot d\mathbf{s}$$

for any two simple, piecewise C^1, oriented curves in the domain of $\mathbf{F}$ with the same endpoints.

Path-Independent Property

Theorem: Let $\mathbf{F}$ be continuous vector field. Then $\mathbf{F}$ has a path-independent line integrals if and only if

$$\oint_C \mathbf{F} \cdot d\mathbf{s} = 0$$

for every piecewise C^1, simple, closed curves C in the domain of $\mathbf{F}$.

Fundamental Theorem of Line Integrals

Suppose that C be a C^1 oriented path given by $\mathbf{c}(t)$, $a \leq t \leq b$. And suppose that f is a function whose gradient vector, ∇f, is continuous on C. Then,

$$\boxed{\int_{\mathbf{c}} \nabla f \cdot d\mathbf{s} = f(\mathbf{c}(b)) - f(\mathbf{c}(b))}$$

Note that $\mathbf{c}(a)$ represents the initial point on C while $\mathbf{c}(b)$ represents the final point on C.

Simply-Connected

A region R in $\mathbb{R}^2$ or $\mathbb{R}^3$ is **simply-connected** if it consists of a single connected piece and if every simple closed curve C in R can be continuously shrunk to a point while remaining in R throughout the deformation.

Test for a vector field to be conservative

Let $\mathbf{F}$ be a vector field of class C^1 whose domain is simply-connected region R in either $\mathbb{R}^2$ or $\mathbb{R}^3$. Then $\mathbf{F} = \nabla f$ for some scalar-valued function f of class C^2 on R if and only if

$$\nabla \times \mathbf{F} = \mathbf{0}$$

for all points of R.

Path-Independence and Conservative fields

If $\mathbf{F}$ is a continuous vector field on an open connected region D and if $\int_C \mathbf{F} \cdot d\mathbf{s}$ is independent of path (for any path in D) then $\mathbf{F}$ is a conservative vector field on D.