Introduction to vectors

January 4, 2006

Vectors in $\mathbb{R}^n$

In this class a **scalar** is simply a real number. An element in $\mathbb{R}$.

A **vector** in $\mathbb{R}^2$ is a pair (x, y) of real numbers.

A **vector** in $\mathbb{R}^3$ is a triple (x, y, z) of real number.

A **vector** in $\mathbb{R}^n$ is an n-tuple $(x_1, x_2, \ldots, x_n)$ of n real numbers.

Operations on vectors

Vector Addition: Let $\mathbf{a} = (a_1, a_2, \ldots, a_n)$ and $\mathbf{b} = (b_1, b_2, \ldots, b_n)$ in $\mathbb{R}^n$ then their **sum** is

$$\mathbf{a} + \mathbf{b} = (a_1 + b_1, a_2 + b_2, \ldots, a_n + b_n)$$

Scalar Multiplication: Let $\mathbf{a} = (a_1, a_2, \ldots, a_n)$ be a vector in $\mathbb{R}^n$ and k any scalar then

$$k\mathbf{a} = (ka_1, ka_2, \ldots, ka_n)$$

The standard basis vectors

The standard basis vectors in $\mathbb{R}^2$ are
$\mathbf{i} = (1, 0)$ and $\mathbf{j} = (0, 1)$.

The standard basis vectors in $\mathbb{R}^3$ are
$\mathbf{i} = (1, 0, 0)$ and $\mathbf{j} = (0, 1, 0)$ and $\mathbf{k} = (0, 0, 1)$.

The standard basis vectors in $\mathbb{R}^n$ are
$\mathbf{e}_1 = (1, 0, \ldots, 0)$, $\mathbf{e}_2 = (0, 1, 0, \ldots, 0), \ldots,$
$\mathbf{e}_n = (0, \ldots, 0, 1)$.

The Dot Product

Let $\mathbf{a} = (a_1, \ldots, a_n)$ and $\mathbf{b} = (b_1, \ldots, b_n)$ be two vectors in $\mathbb{R}^n$. The **dot product** of $\mathbf{a}$ and $\mathbf{b}$ is

$$\mathbf{a} \cdot \mathbf{b} = a_1 b_1 + a2 b_2 + \ldots + a_n b_n$$

When $n = 3$, $\mathbf{a} = (a_1, a_2, a_3)$ and $\mathbf{b} = (b_1, b_2, b_3)$ and $\mathbf{a} \cdot \mathbf{b} = a_1 b_1 + a_2 b_2 + c_1 c_2$.

Length, Angle and Projection

The **length** of a vector is $\|\mathbf{a}\| = \sqrt{\mathbf{a} \cdot \mathbf{a}}$

The **angle** between two vectors $\mathbf{a}$ and $\mathbf{b}$ is $\theta = cos^{-1}(\frac{\mathbf{a} \cdot \mathbf{b}}{\|\mathbf{a}\|\|\mathbf{b}\|})$.

The **projection of vector** $\mathbf{b}$ **onto** $\mathbf{a}$ is $\text{proj}_{\mathbf{a}}\mathbf{b} = (\frac{\mathbf{a} \cdot \mathbf{b}}{\mathbf{a} \cdot \mathbf{a}})\mathbf{a}$

$\frac{\mathbf{a} \cdot \mathbf{b}}{\|\mathbf{a}\|}$ is called the **scalar projection**.

The Cross Product for vectors in $\mathbb{R}^3$

For two vectors $\mathbf{a}$ and $\mathbf{b}$ in $\mathbb{R}^3$ then the **cross product** of $\mathbf{a}$ and $\mathbf{b}$ is the vector $\mathbf{a} \times \mathbf{b}$ such that:

• The length is $\|\mathbf{a} \times \mathbf{b}\| = \|\mathbf{a}\|\|\mathbf{b}\| sin\theta$.

• The direction is determined by extending the fingers of your right hand along the vector $\mathbf{a}$ and curling them towards the vector $\mathbf{b}$, the thumb points in the direction of $\mathbf{a} \times \mathbf{b}$

Note: If $\mathbf{a}$ is parallel to $\mathbf{b}$, then $\mathbf{a} \times \mathbf{b} = \mathbf{0}$.

Determinants

Recall that a matrix is an array of numbers (in our case of real numbers).

The **determinant of a** 2×2 matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is $\det(A) = |A| = ad - bc$.

The **determinant of a** 3×3 matrix $A = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$ is $\det(A) = |A| = aei + bfg + cdh - ceg - afh - bdi$.

Equation of a plane

A plane in $\mathbb{R}^3$ is determined by a point in the plane $P_0(x_0, y_0, z_0)$ and a vector $\mathbf{n} = (A, B, C)$ that is perpendicular (normal) to the plane.

$$\begin{aligned} \mathbf{n} \cdot \vec{P_0P} &= (A, B, C) \cdot (x - x_0, y - y_0, z - z_0) \\ &= A(x - x_0) + B(y - y_0) + C(z - z_0) = 0 \end{aligned}$$

Operations on Matrices

An $m \times n$ **matrix** is an array of real numbers with m rows and n columns.

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} = (a_{ij})$$

The **sum** of two $m \times n$ matrices A and B is the $m \times n$ matrix C obtained by adding the corresponding entries in A and in B, that is $C = A + B = (a_{ij} + b_{ij})$.

Matrix Multiplication

If A is an $m \times n$ matrix and B is an $n \times p$ matrix then the **product** AB is the matrix where the ij-th entry is obtained by taking the dot product of the i-th row of A with the j-th column of B.

NOTE: In order to define the product of A and B we require that the number of columns of A be equal to the number or rows of B. Otherwise, the product is undefined.