1. (15) Given the function $f(x,y,z) = (e^{x-z}, x + \sin(x+z))$ and a function $g(u,v)$ such that

$$(Dg)(u,v) = \begin{pmatrix} e^u & 0 \\ -1 & 1 \\ -ve^{-uv} & -ue^{-uv} \end{pmatrix}$$

and $g(0,0) = (1,0,1)$, find the matrix $D(f \circ g)(0,0)$.

$$Df(x,y,z) = \begin{pmatrix} e^{x-z} & 0 & -e^{x-z} \\ 1+\cos(x-z) & 0 & \cos(x+z) \end{pmatrix}$$

$$Dg(0,0) = \begin{pmatrix} 1 & 0 \\ -1 & 1 \\ 0 & 0 \end{pmatrix}$$

$$D(f \circ g)(0,0) = Df(g(0,0)) \cdot Dg(0,0)$$

$$= Df(1,0,1) \cdot Dg(0,0) \qquad g(0,0) = (1,0,1)$$

so, $$D(f \circ g)(0,0) = \underbrace{\begin{pmatrix} 1 & 0 & -1 \\ 1+\cos(2) & 0 & \cos(2) \end{pmatrix}}_{Df(1,0,1)} \cdot \underbrace{\begin{pmatrix} 1 & 0 \\ -1 & 1 \\ 0 & 0 \end{pmatrix}}_{Dg(0,0)}$$

$$= \begin{pmatrix} 1 & 0 \\ 1+\cos(2) & 0 \end{pmatrix}$$

2. (15) Consider the function $f(x, y) = e^{2x+3y}$.

(i) Find the equation of the tangent plane to the graph of $f(x, y)$ at $(0, 0, 1)$.

$$\frac{\partial f}{\partial x} = 2e^{2x+3y} \qquad \frac{\partial f}{\partial x}(0,0) = 2$$

$$\frac{\partial f}{\partial y} = 3e^{2x+3y} \qquad \frac{\partial f}{\partial y}(0,0) = 3$$

$$2(x-0) + 3(y-0) - (z-1) = 0$$

$$2x + 3y - z = -1$$

(ii) Find the maximum rate of increase of the function $f(x, y)$ at $(0, 0)$ and the direction in which it occurs. (The direction should be a unit vector.)

$$(\nabla f)(0,0) = (2,3) \quad |\nabla f(0,0)| = \sqrt{13}$$ max rate

$$\frac{(\nabla f)(0,0)}{|\nabla f(0,0)|} = \left(\frac{2}{\sqrt{13}}, \frac{3}{\sqrt{13}}\right)$$ direction

(iii) A certain unit vector $\mathbf{u}$ makes an angle of $\frac{\pi}{3}$ with $\nabla f(0, 0)$. Find the directional derivative $(D_{\mathbf{u}}f)(0, 0)$.

$$(D_u f)(0,0) = |\nabla f(0,0)|\,|u|\,\cos\frac{\pi}{3}$$

$$= \frac{\sqrt{13}}{2}$$

could leave

$$\sim \sqrt{13}\cos\frac{\pi}{3}$$

3. (15) Consider the vector field $\mathbf{F}(x,y,z) = (x,y,z)$ and the scalar field $f(x,y,z) = (x^2+y^2+z^2)^{\frac{1}{2}}$.

(i) Compute $\operatorname{curl}\mathbf{F}$.

$$\operatorname{curl} \underline{F} \quad \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & z \end{vmatrix} = \underline{i}\cdot 0 - \underline{j}\,0 + \underline{k}\,0 = (0,0,0)$$

(ii) Compute ∇f and express your answer as a scalar multiple of the vector field $\mathbf{F}$.

$$\nabla f = \left(x(x^2+y^2+z^2)^{-1/2},\ y(x^2+y^2+z^2)^{-1/2},\ z(x^2+y^2+z^2)^{-1/2}\right)$$

$$= \frac{1}{\sqrt{x^2+y^2+z^2}}(x,y,z) = \frac{1}{\sqrt{x^2+y^2+z^2}}\underline{F}$$

(iii) Compute $\operatorname{div}(f\mathbf{F})$ and express your answer as a constant multiple of f.

$$(f\underline{F})(x,y,z) = \left(x(x^2+y^2+z^2)^{1/2},\ y(x^2+y^2+z^2)^{1/2},\ z(x^2+y^2+z^2)^{1/2}\right)$$

$$\operatorname{div} f\underline{F} = (x^2+y^2+z^2)^{1/2} + x^2(x^2+y^2+z^2)^{-1/2} + (x^2+y^2+z^2)^{1/2} + y^2(x^2+y^2+z^2)^{-1/2}$$

$$+ (x^2+y^2+z^2)^{1/2} + z^2(x^2+y^2+z^2)^{-1/2} =$$

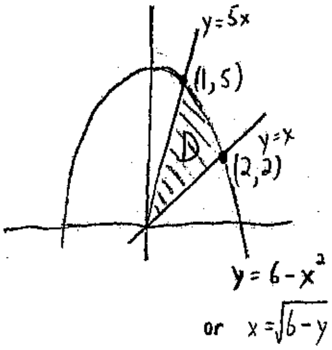
$$3(x^2+y^2+z^2)^{1/2} + (x^2+y^2+z^2)^{-1/2}(x^2+y^2+z^2)$$

$$= 4(x^2+y^2+z^2) = \underline{4f}$$

4. (20) Evaluate the double integral

$$\iint_D xy\,dA,$$

where D is the region in the first quadrant bounded by the parabola $y = 6 - x^2$ and the lines $y = x$ and $y = 5x$. Do not simplify your numerical answer.



$$\iint_D xy\,dA = \int_{x=0}^{1}\int_{y=x}^{5x} xy\,dy\,dx + \int_{x=1}^{2}\int_{y=x}^{y=6-x^2} xy\,dy\,dx$$

$$= \int_0^1 \tfrac12 x\left((5x)^2 - x^2\right)dx + \int_1^2 \tfrac12 x\left((6-x^2)^2 - x^2\right)dx$$

$$= \int_0^1 12x^3\,dx + \tfrac12\int_1^2 x(36 - 13x^2 + x^4)\,dx$$

$$= \tfrac{12}{4}x^4\Big|_0^1 + \tfrac12\int_1^2 (36x - 13x^3 + x^5)\,dx$$

$$= 3 + \tfrac12\left(18x^2 - \tfrac{13}{4}x^4 + \tfrac16 x^6\right)\Big|_1^2$$

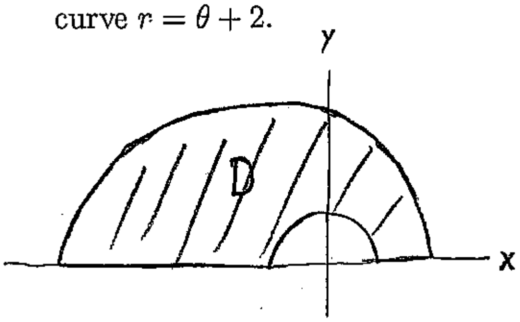
$$= 3 + \tfrac12\left[18(4-1) - \tfrac{13}{4}(2^4-1) + \tfrac16(2^6-1)\right]$$

OR $$\int_{y=0}^{2}\int_{x=\frac{y}{5}}^{x=y} xy\,dx\,dy + \int_{y=2}^{5}\int_{x=\frac{y}{5}}^{x=\sqrt{6-y}} xy\,dx\,dy =$$

5. (15) Evaluate the double integral

$$\int\int_D \theta \, dA,$$

where r and θ are polar coordinates and D is the shaded region pictured below (i.e., D is bounded by (1) the x-axis, (2) the semi-circle of radius 1 and center the origin and (3) the curve $r = \theta + 2$.



$$\int_0^{\pi} \int_1^{\theta+2} \theta r \, dr \, d\theta$$

$$= \int_0^{\pi} \theta \left(\frac{r^2}{2}\right)\Big|_1^{\theta+2} d\theta$$

$$= \frac{1}{2}\int_0^{\pi} \theta \left((\theta+2)^2 - 1^2\right) d\theta$$

$$= \frac{1}{2} \int_0^{\pi} \theta \left(\theta^2 + 4\theta + 4 - 1\right) d\theta$$

$$= \frac{1}{2} \int_0^{\pi} \theta^3 + 4\theta^2 + 3\theta \, d\theta$$

$$= \frac{1}{2}\left(\frac{\theta^4}{4} + \frac{4\theta^3}{3} + \frac{3\theta^2}{2}\right)\Big|_0^{\pi}$$

$$= \frac{1}{2}\left(\frac{\pi^4}{4} + \frac{4\pi^3}{3} + \frac{3\pi^2}{2}\right)$$

$$= \frac{\pi^4}{8} + \frac{2\pi^3}{3} + \frac{3\pi^2}{4}$$

6. (20) Express the integral

$$\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\int_{0}^{y+1} f(x,y,z)dz\,dy\,dx$$

in the form

$$\int\int\int f(x,y,z)\,dx\,dz\,dy.$$

You only need to write in the limits of integration.

$$\int_{-1}^{1}\int_{0}^{y+1}\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} f(x,y,z)\,dx\,dz\,dy$$