Math 13 Homework #9
Will not be collected

(1) Let $\mathbf{F} = \langle y, x, x^2 + y^2 \rangle$ and let $\mathcal{S}$ be the top half of the sphere $x^2 + y^2 + z^2 = 1$ (i.e., $z \geq 0$), oriented with outward-pointing normal vectors. Let $\mathcal{C}$ be the circle $x^2 + y^2 = 1$ in the xy-plane with the boundary orientation. Verify Stokes' Theorem by computing

$$\oint_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r} \qquad \text{and} \qquad \iint_{\mathcal{S}} \operatorname{curl}(\mathbf{F}) \cdot d\mathbf{S}.$$

(2) Let $\mathbf{F} = \langle y^2, 2z + x, 2y^2 \rangle$. Use Stokes' Theorem to find a plane with equation $ax + by + cz = 0$ (where a, b, and c are not all zero) such that $\oint_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r} = 0$ for every closed curve $\mathcal{C}$ lying in the plane. (Hint: choose a, b, c so that $\operatorname{curl}(\mathbf{F})$ lies in the plane).

(3) Let I be the flux of $\mathbf{F} = \langle e^y, 2xe^{x^2}, z^2 \rangle$ out of the upper hemisphere $\mathcal{S}$ of the unit sphere.

(a) Let $\mathbf{G} = \langle e^y, 2xe^{x^2}, 0 \rangle$. Find a vector field $\mathbf{A} = \langle 0, 0, A_3 \rangle$ such that $\operatorname{curl}(\mathbf{A}) = \mathbf{G}$.

(b) Use Stokes' Theorem to show that the flux of $\mathbf{G}$ through $\mathcal{S}$ is zero (i.e., show that $\iint_{\mathcal{S}} \mathbf{G} \cdot d\mathbf{S} = 0$.)

(c) Calculate I. *Hint:* Use part (b) to show that $I = \iint_{\mathcal{S}} \langle 0, 0, z^2 \rangle \cdot d\mathbf{S}$.

(4) Use the Divergence Theorem to evaluate the flux $\iint_{\mathcal{S}} \mathbf{F} \cdot d\mathbf{S}$, where $\mathbf{F} = \langle y, z, x \rangle$ and $\mathcal{S}$ is the sphere $x^2 + y^2 + z^2 = 1$, oriented with outward-pointing normal vectors.

(5) Let $\mathcal{W}$ be the region between the sphere of radius 4 and the cube of side lengths 1, both centered at the origin. What is the flux though the boundary $\mathcal{S} = \partial \mathcal{W}$ of a vector field $\mathbf{F}$ whose divergence has the constant value $\operatorname{div}(\mathbf{F}) = -4$?