Math 13 Homework #8
Due Wednesday, May 22nd

(1) Evaluate the following vector surface integrals

$$\iint_{\mathcal{S}} \mathbf{F} \cdot d\mathbf{S}$$

for the given vector field $\mathbf{F}$ and surface $\mathcal{S}$.

(a) $\mathbf{F} = \langle e^z, z, x \rangle$, $\mathcal{S}$ is given by $G(r, s) = (rs, r+s, r), 0 \leq r \leq 1, 0 \leq s \leq 1$, oriented by $\mathbf{T}_r \times \mathbf{T}_s$.

(b) $\mathbf{F} = y^2\mathbf{i} + 2\mathbf{j} - x\mathbf{k}$, $\mathcal{S}$ is the portion of the plane $x + y + z = 1$ in the octant $x, y, z \geq 0$, and oriented with upward-pointing normal vectors.

(2) Verify Green's Theorem for the following line integrals, by evaluating the integral in two different ways: directly, as a line integral over the specified curve $\mathcal{C}$, and also as the double integral over the region $\mathcal{D}$ bounded by $\mathcal{C}$.

(a) $\oint_{\mathcal{C}} xydx + ydy$, where $\mathcal{C}$ is the unit circle, oriented clockwise.

(b) $\oint_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F}(x, y) = \langle x + y, x^2 - y \rangle$ and $\mathcal{C}$ is the boundary of the region enclosed by $y = x^2$ and $y = \sqrt{x}$ for $0 \leq x \leq 1$.

(3) Compute the area of the following regions in $\mathbb{R}^2$ by writing it as the integral over the boundary.

(a) The triangle with vertices $(0, 0)$, $(1, 1)$, $(1, 2)$.

(b) The ellipse given by $\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$.

(4) Let $\mathbf{F}$ be the vortex field

$$\mathbf{F} = \left\langle \frac{-y}{x^2 + y^2}, \frac{x}{x^2 + y^2} \right\rangle.$$

Show that

$$\oint_{\mathcal{C}_1} \mathbf{F} \cdot d\mathbf{r} = \oint_{\mathcal{C}_2} \mathbf{F} \cdot d\mathbf{r},$$

where $\mathcal{C}_1$ is the circle of radius R_1 and $\mathcal{C}_2$ is the circle of radius R_2, both centered at the origin, both oriented counterclockwise (Hint: Apply the general form of Green's Theorem to the region between $\mathcal{C}_1$ and $\mathcal{C}_2$.)