

Math 13 Spring 2017, Midterm Exam II
Elements of solution

(1) For each of the following assertions, select the correct ending.

(a) Let \mathbb{D} be the disk with radius 1 and center at the origin in the uv -plane and G the mapping defined by

$$G(u, v) = (u + 2v, -u + 4v).$$

The area of $G(\mathbb{D})$ is...

6π .

$\frac{\pi}{6}$.

π .

2π .

$\frac{\pi}{2}$.

none of the above.

(b) Let $f(x, y, z) = y^3$. Then $\text{grad}(\text{div}(\text{grad } f)) = \dots$

$6\mathbf{j}$.

6 .

$6y$.

$\langle 0, 3y^2, 0 \rangle$.

none of the above.

(c) Let \mathbf{F} be a vector field. Then $\text{grad}(\text{curl } \mathbf{F})$ is...

a potential for \mathbf{F} .

the divergence of \mathbf{F} .

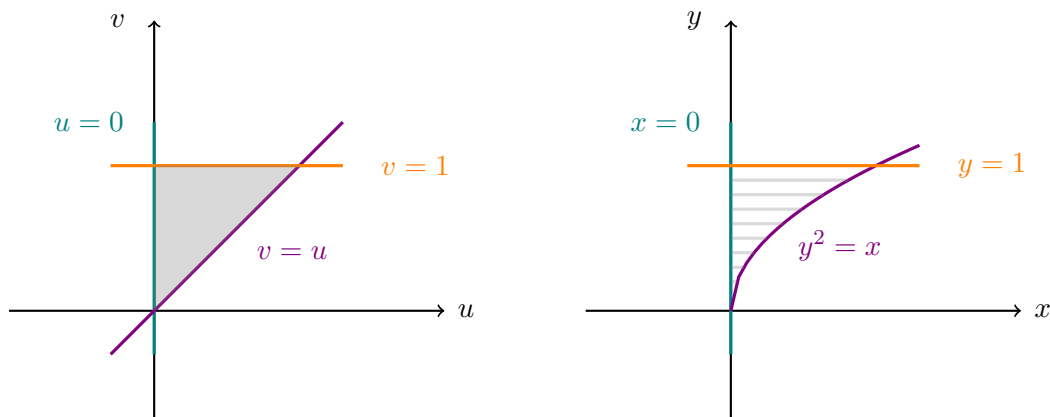
a vector field.

not well-defined.

none of the above.

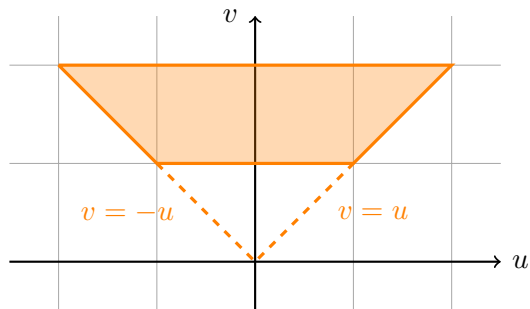
- (2) Determine the image of the triangular region with vertices $(0,0)$, $(1,1)$ and $(0,1)$ under the transformation $G(u,v) = (u^2, v)$.

The transformation is given by $x = u^2$ and $y = v$ and the original region is bounded by the lines with equations $u = 0$, $v = 1$ and $u = v$ which transform under G to $x = 0$, $y = 1$ and $y^2 = x$.



- (3) The mapping $G(u,v) = \left(\frac{1}{2}(u+v), \frac{1}{2}(u-v)\right)$ transforms the trapezoidal region S with vertices $(-2,2)$, $(2,2)$, $(1,1)$ and $(-1,1)$ in the uv -plane into the trapezoidal region \mathcal{R} with vertices $(1,0)$, $(2,0)$, $(0,-2)$ and $(0,-1)$ in the xy -plane.

Use this change of variables to evaluate the integral $\iint_{\mathcal{R}} e^{\frac{x+y}{x-y}} dA$.



Observing that $x + y = u$ and $x - y = v$ and that $\text{Jac}(G) = -\frac{1}{2}$, we obtain

$$\begin{aligned}
 \iint_{\mathcal{R}} e^{\frac{x+y}{x-y}} dA &= \frac{1}{2} \iint_S e^{\frac{u}{v}} du dv \\
 &= \frac{1}{2} \int_1^2 \int_{-v}^v e^{\frac{u}{v}} du dv \\
 &= \frac{1}{2} \int_1^2 v \left(e - \frac{1}{e} \right) dv \\
 &= \boxed{\frac{3}{4} \left(e - \frac{1}{e} \right)}.
 \end{aligned}$$

- (4) **The volume of the solid ellipsoid \mathcal{E} :** $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \leq 1$ **is given by:**

$$\text{Vol}(\mathcal{E}) = \iiint_{\mathcal{E}} 1 \, dV.$$

Use one of the following changes of variables and the fact that the ball with center at the origin and radius 1 has volume $\frac{4}{3}\pi$ to determine $\text{Vol}(\mathcal{E})$.

$$G_1(u, v, w) = \left(\frac{u}{a}, \frac{v}{b}, \frac{w}{c} \right)$$

$$G_2(u, v, w) = (au, bv, cw)$$

$$G_3(u, v, w) = (\sqrt{u}, \sqrt{v}, \sqrt{w})$$

Let us use G_2 and set $x = au$, $y = bv$ and $z = cw$. Then,

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \leq 1 \quad \Leftrightarrow \quad u^2 + v^2 + w^2 \leq 1$$

and the Change of Variables Formula gives

$$\text{Vol}(\mathcal{E}) = \iiint_{\mathcal{B}} 1 \cdot |\text{Jac}(G_2)| \, du \, dv \, dw$$

where \mathcal{B} denotes the unit ball in the uvw -space and, assuming $a, b, c > 0$,

$$\text{Jac}(G_2) = \begin{vmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{vmatrix} = abc.$$

It follows that

$$\text{Vol}(\mathcal{E}) = abc \iiint_{\mathcal{B}} 1 \, du \, dv \, dw = abc \cdot \text{Vol}(\mathcal{B}) = \boxed{\frac{4abc}{3}\pi}.$$

- (5) **Evaluate the integral**

$$\int_{\mathcal{C}} 18y^3 \, ds$$

where \mathcal{C} is the curve in the plane parameterized by $x(t) = t^3$, $y(t) = t$ with $0 \leq t \leq 1$.

By definition,

$$\begin{aligned} \int_{\mathcal{C}} 18y^3 \, ds &= 18 \int_0^1 y(t)^3 \sqrt{x'(t)^2 + y'(t)^2} \, dt \\ &= 18 \int_0^1 t^3 \sqrt{9t^4 + 1} \, dt. \end{aligned}$$

The change of variables $u = 9t^4 + 1$ with $du = 36t^3 \, dt$ further leads to

$$\int_{\mathcal{C}} 18y^3 \, ds = \frac{1}{2} \int_1^{10} \sqrt{u} \, du = \boxed{\frac{10\sqrt{10} - 1}{3}}.$$

- (6) Evaluate the integral $\int_{\mathcal{L}} xe^{yz} ds$ where \mathcal{L} is the segment from $(0, 0, 0)$ to $(1, 2, 3)$.

We can parametrize \mathcal{L} by $\mathbf{r}(t) = \langle t, 2t, 3t \rangle$ for $0 \leq t \leq 1$ so that $ds = \sqrt{14} dt$ and

$$\int_{\mathcal{L}} xe^{yz} ds = \sqrt{14} \int_0^1 te^{6t^2} dt = \frac{\sqrt{14}}{12} \int_0^6 e^u du = \boxed{\frac{\sqrt{14}(e^6 - 1)}{12}}.$$

- (7) Evaluate the integral of the vector field

$$\mathbf{F}(x, y, z) = \left(x + y + \frac{z}{4}\right) \mathbf{i} + (y - x^3) \mathbf{j} + \ln\left(\frac{x+z}{y+1}\right) \mathbf{k}$$

along the curve Γ given by $\mathbf{r}(t) = t\mathbf{i} + t^3\mathbf{j} + 4\mathbf{k}$ for $0 \leq t \leq 1$.

$$\begin{aligned} \int_{\Gamma} \mathbf{F} \cdot d\mathbf{r} &= \int_0^1 \mathbf{F}(x(t), y(t), z(t)) \cdot \mathbf{r}'(t) dt \\ &= \int_0^1 \left\langle t + t^3 + 1, 0, \ln\left(\frac{t+4}{t^3+1}\right) \right\rangle \cdot \langle 1, 3t^2, 0 \rangle dt \\ &= \int_0^1 (t + t^3 + 1) dt = \boxed{\frac{7}{4}}. \end{aligned}$$

- (8) The vector field $\mathbf{F}(x, y) = (3+2xy)\mathbf{i} + (x^2-3y^2)\mathbf{j}$ is conservative. Find a potential.

A potential for \mathbf{F} is a function $f(x, y)$ satisfying

$$\frac{\partial f}{\partial x}(x, y) = 3 + 2xy \quad \text{and} \quad \frac{\partial f}{\partial y}(x, y) = x^2 - 3y^2.$$

The first identity implies that

$$f(x, y) = 3x + x^2y + K(y)$$

where $K(y)$ is a function of y only. Differentiating this identity with respect to y gives

$$\frac{\partial f}{\partial y}(x, y) = x^2 + K'(y)$$

and the second potential relation implies that $K'(y) = -3y^2$, so that $K(y) = -y^3 + C$ where C is any real constant. As a conclusion, any potential for \mathbf{F} is of the form

$$\boxed{3x + x^2y - y^3 + C}.$$

- (9) Let $f(x, y) = xe^y$ and $\mathbf{F} = \nabla f$. Evaluate $\int_{\Gamma} \mathbf{F} \cdot d\mathbf{r}$, where Γ is given by

$$\mathbf{r}(t) = te^t\mathbf{i} + \sqrt{1+3t}\mathbf{j}$$

for $0 \leq t \leq 1$.

The Fundamental Theorem of Calculus for line integrals gives

$$\int_{\Gamma} \mathbf{F} \cdot d\mathbf{r} = f(\mathbf{r}(1)) - f(\mathbf{r}(0)) = f(e, 2) - f(0, 1) = \boxed{e^3}.$$