

Math 13 Spring 2017, Midterm Exam I
Elements of solution

(1) Let f be a function of three variables x , y and z . For each of the following assertions, select the correct answer.

(a) The integral

$$\int_0^{2t} \int_0^x \int_0^{xz} f(x, y, z) dy dz dx$$

- is a constant.
- is a function of t .
- is a function of y .
- is not well-defined.
- none of the above.

(b) The integral

$$\int_0^1 \int_x^0 \int_0^{xz} f(x, y, z) dy dx dz.$$

- is a constant.
- is a function of x and z .
- is a function of y .
- is not well-defined.
- none of the above.

(c) The integral

$$\int_0^1 \int_0^2 f(x, y, z) dx dz.$$

- is a constant.
- is a function of x and z .
- is a function of y .
- is not well-defined.
- none of the above.

(2) Calculate the double integral

$$\iint_{\mathcal{R}} \left(\frac{x}{y} + \frac{y}{x} \right) dA$$

where $\mathcal{R} = [1, 4] \times [1, 2]$.

$$\begin{aligned} \iint_{\mathcal{R}} \left(\frac{x}{y} + \frac{y}{x} \right) dA &= \int_{y=1}^2 \int_{x=1}^4 \left(\frac{x}{y} + \frac{y}{x} \right) dx dy \\ &= \int_{y=1}^2 \left[\frac{1}{y} \frac{x^2}{2} + y \ln x \right]_{x=1}^4 dy \\ &= \int_{y=1}^2 \left(\frac{15}{2y} + 2y \ln 2 \right) dy \\ &= \left[\frac{15}{2} \ln y + y^2 \ln 2 \right]_{y=1}^2 = \boxed{\frac{21 \ln 2}{2}}. \end{aligned}$$

(3) Evaluate the integral $\int_0^1 \int_{3y}^3 e^{x^2} dx dy$.

No antiderivative of e^{x^2} can be expressed in terms of ordinary functions so we reverse the order of integration:

$$\begin{aligned} \int_0^1 \int_{3y}^3 e^{x^2} dx dy &= \int_0^3 \int_0^{\frac{x}{3}} e^{x^2} dy dx \\ &= \int_0^3 \left[ye^{x^2} \right]_0^{\frac{x}{3}} dx \\ &= \int_0^3 \frac{x}{3} e^{x^2} dx \\ &= \frac{1}{3} \int_{u=0}^9 e^u \frac{du}{2} = \boxed{\frac{e^9 - 1}{6}}. \end{aligned}$$

(4) Calculate the triple integral $\int_0^{\sqrt{\pi}} \int_0^x \int_0^{xz} x^2 \cos(y) dy dz dx$.

$$\begin{aligned} \int_0^{\sqrt{\pi}} \int_0^x \int_0^{xz} x^2 \cos(y) dy dz dx &= \int_0^{\sqrt{\pi}} \int_0^x x^2 \sin(xz) dz dx \\ &= \int_0^{\sqrt{\pi}} [-x \cos(xz)]_{z=0}^x dx \\ &= \int_0^{\sqrt{\pi}} (-x \cos(x^2) + x) dx \\ &= -\int_0^{\pi} \cos(u) \frac{du}{2} + \frac{\pi}{2} = \boxed{\frac{\pi}{2}}. \end{aligned}$$

- (5) **Find the volume of the solid that lies below the surface $z^2 = 9x^2 + 9y^2$, above the xy -plane, and inside the cylinder $x^2 + y^2 = 2y$.**

The base of the cylinder is the circle in the xy -plane with center $(0, 1)$ and radius 1. In polar coordinates, it is described by $r = 2 \sin \theta$ for $0 \leq \theta \leq \pi$. Since z is assumed non-negative, the upper boundary of the solid has equation $z = 3r$ in cylindrical coordinates.

It follows that

$$\begin{aligned} \text{Volume} &= \int_0^\pi \int_0^{2 \sin \theta} \int_0^{3r} r \, dz \, dr \, d\theta \\ &= \int_0^\pi \int_0^{2 \sin \theta} 3r^2 \, dr \, d\theta \\ &= 8 \int_0^\pi \sin^3 \theta \, d\theta = \boxed{\frac{32}{3}}. \end{aligned}$$

- (6) **The volume of the solid bounded by the surfaces with equations**

$$y = x^2 + \frac{z^2}{4} \quad \text{and} \quad y = 5 - 4x^2 - z^2$$

is given by an integral of the form

$$\int_a^b \int_{u_1(z)}^{u_2(z)} \int_{x^2 + \frac{z^2}{4}}^{5 - 4x^2 - z^2} dy \, dx \, dz.$$

Determine $u_1(z)$, $u_2(z)$, a and b .

To identify the projection onto the xz -plane of the solid, we determine the intersection curve of the paraboloids:

$$y = x^2 + \frac{z^2}{4} = 5 - 4x^2 - z^2$$

implies $x^2 + \frac{z^2}{4} = 1$, which is the equation of an ellipse in the $y = 0$ plane. It follows that

$$a = -2 \quad , \quad b = 2 \quad , \quad u_1(z) = -\sqrt{1 - \frac{z^2}{4}} \quad , \quad u_2(z) = \sqrt{1 - \frac{z^2}{4}}.$$

(7) Let \mathcal{W} be the solid region lying:

- inside the cone $z = \sqrt{x^2 + y^2}$
- inside the sphere $x^2 + y^2 + z^2 = 5$
- outside the sphere $x^2 + y^2 + z^2 = 3$
- in the second octant: $x \leq 0$, $y \geq 0$, $z \geq 0$.

Convert the integral $\iiint_{\mathcal{W}} x \, dV$ into a triple integral in spherical coordinates.

In spherical coordinates, \mathcal{W} consists of the points (ρ, θ, φ) such that

$$\sqrt{3} \leq \rho \leq \sqrt{5} \quad , \quad \frac{\pi}{2} \leq \theta \leq \pi \quad , \quad 0 \leq \varphi \leq \frac{\pi}{4}$$

Therefore,

$$\iiint_{\mathcal{W}} x \, dV = \int_{\frac{\pi}{2}}^{\pi} \int_0^{\frac{\pi}{4}} \int_{\sqrt{3}}^{\sqrt{5}} \rho^3 \sin^2 \varphi \cos \theta \, d\rho \, d\varphi \, d\theta.$$

(8) Express the volume of the region bounded by the surfaces with equations

- $z = x^2 - 1$
- $x = y^2$
- $z = 0$

by a triple integral in the prescribed order:

- (a) $dz \, dy \, dx$
(b) $dy \, dx \, dz$

$$\text{Volume} = \int_{x=0}^1 \int_{y=-\sqrt{x}}^{\sqrt{x}} \int_{z=x^2-1}^0 dz \, dy \, dx = \int_{z=-1}^0 \int_{x=0}^{\sqrt{z+1}} \int_{y=-\sqrt{x}}^{\sqrt{x}} dy \, dx \, dz.$$

(9) A lamina has the shape of a half-disk of radius 2, centered at the origin and lying in the upper-half plane. Its density function is $\delta(x, y) = k\sqrt{x^2 + y^2}$, where k is a constant. The center of mass of the lamina has coordinates $(0, y_0)$. Determine y_0 .

Observe that the density only depends on the distance to the origin: $\delta(r, \theta) = kr$. The total mass of the lamina is

$$M = \int_0^{\pi} \int_0^2 kr^2 \, dr \, d\theta = \frac{8k\pi}{3}.$$

The vertical moment is

$$M_x = \int_0^{\pi} \int_0^2 kr^3 \sin \theta \, dr \, d\theta = 8k.$$

It follows that

$$y_0 = \frac{M_x}{M} = \frac{3}{\pi}.$$