Homework 6

Elements of solution

Problem 1: **Find an equation of the tangent plane to the surface parametrized by $\mathbf{S}(u,v) = (u+v)\mathbf{i} + 3u^2\mathbf{j} + (u-v)\mathbf{k}$ at $(2,3,0)$.**

The point $(2,3,0)$ is obtained for the values $u_0 = 1$ and $v_0 = 1$ of the parameters. Tangents vectors to the surface at that point are therefore given by:

$$\mathbf{T}_u(u_0,v_0) = \langle 1, 6u_0, 1\rangle = \langle 1,6,1\rangle \qquad \text{and} \qquad \mathbf{T}_v(u_0,v_0) = \langle 1,0,-1\rangle$$

and an equation of the plane through $(2,3,0)$ with normal vector

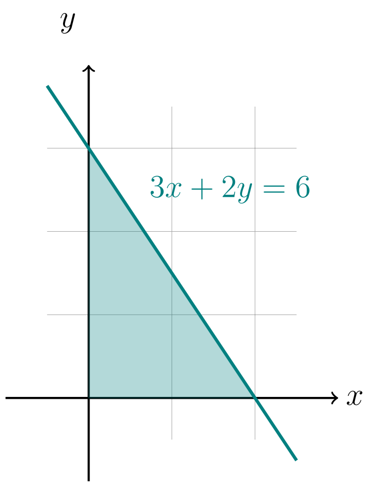
$$\langle 1,6,1\rangle \times \langle 1,0,-1\rangle = \langle -6,2,-6\rangle$$

is $\boxed{3x - y + 3z - 3 = 0}$.

Problem 2: **Evaluate the area of:**

(a) **the part of the plane $3x + 2y + z = 6$ that lies in the first octant;**

A normal vector (at every point) of this plane is $\langle 3,2,1\rangle$. Its intersection with the xy-plane in the first octant is bounded by the lines with equation $x = 0$, $y = 0$ and $3x + 2y = 6$.



The area of the part of the plane above this triangle is therefore given by

$$\int_0^3 \int_0^{2-\frac{2}{3}y} \| \langle 3, 2, 1 \rangle \| \, dx\, dy = \sqrt{14} \int_0^3 \left(2 - \frac{2}{3} y \right) \, dy = \boxed{3\sqrt{14}}.$$

(b) **the part of the cone $x^2 + y^2 = z^2$ that lies in the half-space $x < 0$ between heights $z = 1$ and $z = 2$.**

This part of the cone can be parametrized by

$$S(u, v) = (u \cos v, u \sin v, u)$$

with $1 \leq u \leq 2$ and $\frac{\pi}{2} \leq v \leq \frac{3\pi}{2}$. Tangent vectors at the point $S(u, v)$ are

$$\mathbf{T}_u = \langle \cos v \, , \, \sin v \, , \, 1 \rangle$$

and

$$\mathbf{T}_v = \langle -u \sin v \, , \, u \cos v \, , \, 0 \rangle$$

so that a normal vector to the cone at the point $S(u, v)$ is

$$\mathbf{N}(u, v) = \mathbf{T}_u \times \mathbf{T}_v = \langle -u \cos v \, , \, -u \sin v, u \rangle \, ,$$

with magnitude $\|\mathbf{N}(u, v)\| = \sqrt{2u^2}$.
The area of the surface at hand is therefore

$$\sqrt{2} \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \int_1^2 |u| \, du \, dv = \pi \sqrt{2} \int_1^2 u \, du = \boxed{\frac{3\pi\sqrt{2}}{2}}.$$

Problem 3: Calculate the integral

$$\iint_\Sigma y^2\, dS$$

where Σ is the part of the sphere $x^2+y^2+z^2=4$ that lies inside the cylinder $x^2+y^2=1$ and above the xy-plane in the $y<0$ region.

The intersection between the sphere and the cylinder in the upper half-space can be described in spherical coordinates by $\varphi = \dfrac{\pi}{6}$, $r=2$. Therefore, Σ can be parametrized by

$$S(\theta,\varphi) = (2\cos\theta\sin\varphi\,,\ 2\sin\theta\sin\varphi\,,\ 2\cos\varphi)$$

with $\pi \le \theta \le 2\pi$ and $0 \le \varphi \le \dfrac{\pi}{6}$, with tangent vectors

$$\mathbf{T}_\theta = \langle -2\sin\theta\sin\varphi\,,\ 2\cos\theta\sin\varphi,\ 0\rangle \quad \text{and} \quad \mathbf{T}_\varphi = \langle 2\cos\theta\cos\varphi\,,\ 2\sin\theta\cos\varphi\,,\ -2\sin\varphi\rangle$$

hence a normal vector

$$\mathbf{N}(\theta,\varphi) = \mathbf{T}_\theta \times \mathbf{T}_\varphi = -4\sin\varphi\frac{\mathbf{r}}{\|\mathbf{r}\|}$$

It follows that

$$\begin{aligned}
\int_\Sigma y^2\, dS &= \int_\pi^{2\pi}\int_0^{\frac{\pi}{6}} (2\sin\theta\sin\varphi)^2 |4\sin\varphi|\, d\varphi\, d\theta \\
&= 16\int_\pi^{2\pi} \sin^2\theta\, d\theta \cdot \int_0^{\frac{\pi}{6}} \sin^3\varphi\, d\varphi \\
&= \boxed{8\pi\left(\frac{2}{3} - \frac{3\sqrt{3}}{8}\right)}.
\end{aligned}$$