Homework 4

Elements of solution

Problem 1:

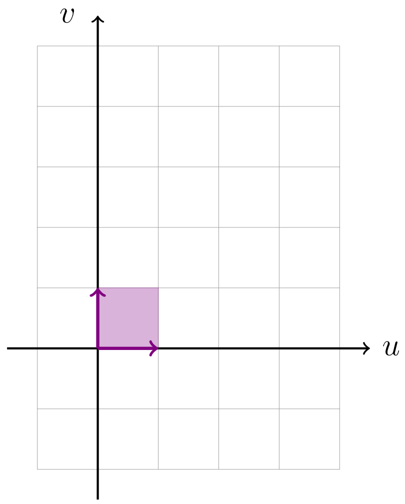
(a) **Find a linear mapping $G(u, v)$ that maps the unit square $[0, 1] \times [0, 1]$ to the parallelogram in the xy-plane spanned by the vectors $\langle 3, -1\rangle$ and $\langle 1, 4\rangle$.**

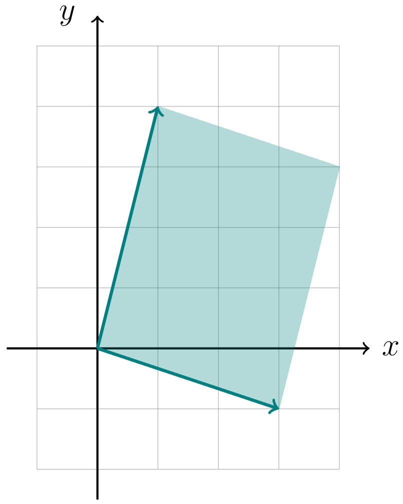
We look for constants A, B, C and D such that the mapping

$$G(u, v) = (Au + Bv, Cu + Dv)$$

satisfies $G(1, 0) = (3, -1)$ and $G(0, 1) = (1, 4)$. These relations immediately give

$$\boxed{A = 3 \quad , \quad B = 1 \quad , \quad C = -1 \quad \text{and} \quad D = 4}.$$





(b) **Use the Jacobian to find the area of the image of the rectangle $\mathcal{R} = [0, 4] \times [0, 3]$ under G.**

The Jacobian of this transformation is $AD - BC = 13$, so that the area of the parallelogram spanned by $\langle 3, -1\rangle$ and $\langle 1, 4\rangle$ is 13 times that of the rectangle $\mathcal{R}$, that is $13 \times 12 = \boxed{156}$.

Problem 2: Use the change of variables $x = u^2 - v^2$, $y = 2uv$ to evaluate the integral

$$\iint_{\mathcal{A}} y \, dA$$

where $\mathcal{A}$ is the region bounded by the x-axis and the parabolas $y^2 = 4 - 4x$ and $y^2 = 4 + 4x$, with $y \geq 0$.

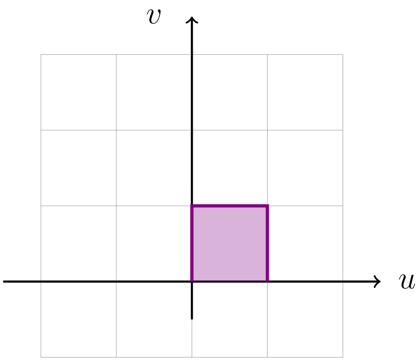
First, we determine the preimage in the uv-plane of $\mathcal{A}$ under the change of variables $G(u, v) = (u^2 - v^2, 2uv)$. Notice that $y \geq 0$ means that x and y have the same sign.

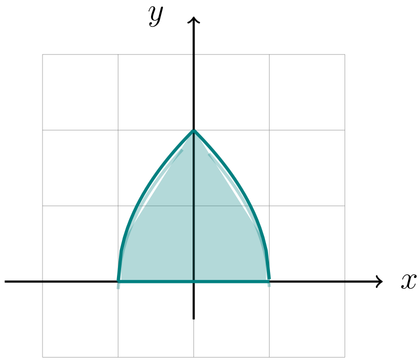
$$\begin{aligned} y^2 = 4 - 4x \quad &\Leftrightarrow \quad 4u^2v^2 = 4 - 4u^2 + 4v^2 \\ &\Leftrightarrow \quad (u^2 - 1)(v^2 + 1) = 0 \\ &\Leftrightarrow \quad u = \pm 1 \end{aligned}$$

Similarly,

$$y^2 = 4 + 4x \Leftrightarrow v = \pm 1$$

so that the preimage of the domain is the square $[0, 1] \times [0, 1]$ in the uv-plane.





The Jacobian of the transformation is

$$\text{Jac}(G) = \begin{vmatrix} 2u & -2v \\ v & u \end{vmatrix} = 4(u^2 + v^2).$$

By the Change of Variables formula, it follows that

$$\begin{aligned} \iint_{\mathcal{A}} y \, dA \quad &= \quad \iint_{[0,1]\times[0,1]} 2uv \cdot 4(u^2 + v^2) \\ &= \quad 8 \int_0^1 \int_0^1 (u^3 v + u v^3) \, du \, dv = \boxed{2}. \end{aligned}$$

Problem 3: **Let $\mathcal{D}$ be the region in the xy-plane bounded by the curves**

$$y = \frac{2}{x} \quad , \quad y = \frac{1}{2x} \quad , \quad y = 2x \quad , \quad y = \frac{x}{2}$$

and F the map from he xy-plane to the uv-plane given by $u = xy$ and $v = \dfrac{y}{x}$.

(a) **Sketch $\mathcal{D}$ and find the image of $\mathcal{D}$ under F.**

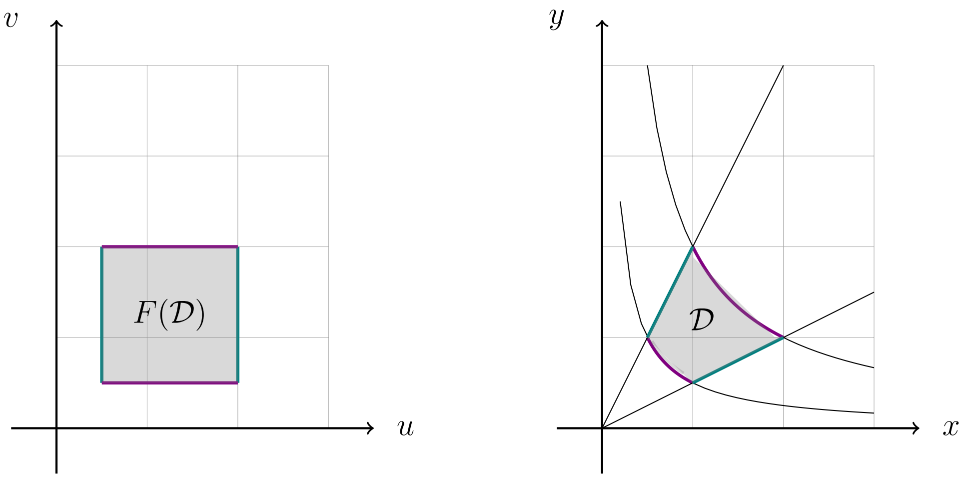
The boundaries of $\mathcal{D}$ have equations

$$xy = \frac{1}{2} \quad , \quad xy = 2 \quad , \quad \frac{y}{x} = \frac{1}{2} \quad \text{and} \quad \frac{y}{x} = 2$$

which respectively translate under F to:

$$u = \frac{1}{2} \quad , \quad u = 2 \quad , \quad v = \frac{1}{2} \quad \text{and} \quad v = 2$$

so that the image of $\mathcal{D}$ under F is the square $\left[\frac{1}{2}, 2\right] \times \left[\frac{1}{2}, 2\right]$ in the uv-plane.



(b) **Let $G = F^{-1}$. Determine $|\text{Jac}(G)|$.**

We know that $\text{Jac}(F^{-1}) = \text{Jac}(F)^{-1}$ so it suffices to calculate

$$\text{Jac}(F) = \frac{\partial(u, v)}{\partial(x, y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial y}{\partial y} \end{vmatrix} = \frac{2y}{x}$$

to conclude that $|\text{Jac}(G)| = \dfrac{x}{2y} = \dfrac{1}{2v}$.

(c) **Apply the Change of Variables formula to find a relation between**

$$\iint_{\mathcal{D}} f\left(\frac{y}{x}\right)\, dA \qquad \textbf{and} \qquad \int_{\frac{1}{2}}^{2} \frac{f(v)}{v}\, dv.$$

Note that f is a function of **one** variable, that the function $(x, y) \mapsto f\left(\frac{y}{x}\right)$ is the composition of f and v, which is a function of **two** variables, and that

$$\iint_{\mathcal{D}} f\left(\frac{y}{x}\right)\, dA = \iint_{\mathcal{D}} f\left(v(x,y)\right)\, dA.$$

The Change of Variables formula, then gives

$$\begin{aligned}
\iint_{\mathcal{D}} f\left(v(x,y)\right)\, dA &= \iint_{\left[\frac{1}{2},2\right]\times\left[\frac{1}{2},2\right]} f(v)\left|\mathrm{Jac}(G)\right|\, du\, dv \\
&= \int_{\frac{1}{2}}^{2}\int_{\frac{1}{2}}^{2} f(v)\frac{1}{2v}\, dv\, du \\
&= \frac{1}{2}\int_{\frac{1}{2}}^{2} du \int_{\frac{1}{2}}^{2} \frac{f(v)}{v}\, dv \\
&= \frac{3}{4}\int_{\frac{1}{2}}^{2} \frac{f(v)}{v}\, dv.
\end{aligned}$$

(d) **Use (c) to evaluate** $\displaystyle\iint_{\mathcal{D}} \frac{ye^{\frac{y}{x}}}{x}\, dA$**.**

The integral at hand is of the form $\displaystyle\iint_{\mathcal{D}} f\left(\frac{y}{x}\right)\, dA$ with $f(t) = te^t$. Using the result of the previous question, we get

$$\iint_{\mathcal{D}} f\left(\frac{y}{x}\right)\, dA = \frac{3}{4}\int_{\frac{1}{2}}^{2} \frac{ve^v}{v}\, dv = \frac{3}{4}\int_{\frac{1}{2}}^{2} e^v\, dv = \boxed{\frac{3(e^2 - \sqrt{e})}{4}}.$$