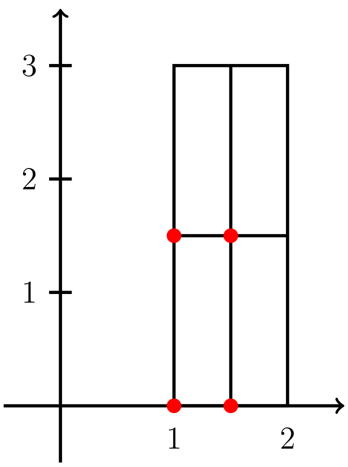
Homework 2

Elements of Solution

Problem 1: Estimate the volume of the solid that lies below the surface $z = 1 + x^2 + 3y$ and above the rectangle $\mathcal{R} = [1, 2] \times [0, 3]$ by a Riemann sum with $N = M = 2$ and sample points the lower left corners. (Draw a picture).

The sample points are: $(1, 0)$, $(1.5, 0)$,$(1, 1.5)$ and $(1.5, 1.5)$ and all rectangles in the subdivision have area

$$\Delta x \cdot \Delta y = 0.5 \times 1.5 = 0.75$$



Let $f(x, y) = 1 + x^2 + 3y$. The estimate for $\iint_{\mathcal{R}} f(x, y)\, dA$ is

$$\begin{aligned} S_{2,2} &= 0.75 \times (f(1, 0) + f(1.5, 0) + f(1, 1.5) + f(1.5, 1.5)) \\ &= 0.75 \times (2 + 3.25 + 6.5 + 7.75) \\ &= 0.75 \times 19.5 \\ &= 14.625. \end{aligned}$$

Problem 2: **Calculate the following two integrals.**

(a) $I_1 = \displaystyle\iint_{\mathcal{R}_1} \frac{xy^2}{x^2+1}\, dA$, **where** $\mathcal{R}_1 = [0,1] \times [-3,3]$

Let us write I_1 as an iterated integral:

$$\begin{aligned}
\iint_{\mathcal{R}_1} \frac{xy^2}{x^2+1}\, dA &= \int_{x=0}^{1}\int_{y=-3}^{3} \frac{xy^2}{x^2+1}\, dy\, dx \\
&= \int_{x=0}^{1} \frac{x}{x^2+1} \int_{y=-3}^{3} y^2\, dy\, dx \qquad \text{since } \frac{x}{x^2+1} \text{ is constant in } y \\
&= 18 \int_{x=0}^{1} \frac{x}{x^2+1}\, dx \qquad \text{since } \int_{y=-3}^{3} y^2\, dy = 18 \\
&= 18 \left[\frac{1}{2}\ln(x^2+1)\right]_{x=0}^{1} \\
&= 9\ln(2).
\end{aligned}$$

(b) $I_2 = \displaystyle\iint_{\mathcal{R}_2} \frac{x}{1+xy}\, dA$, **where** $\mathcal{R}_2 = [0,1] \times [0,1]$

Again, we write I_2 as an iterated integral:

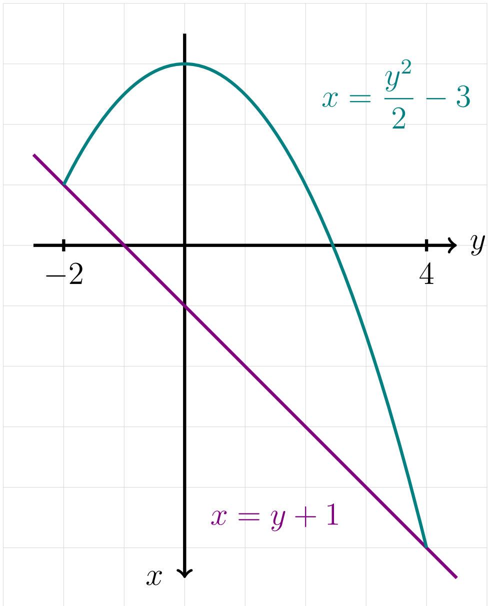
$$\begin{aligned}
\iint_{\mathcal{R}_2} \frac{x}{1+xy}\, dA &= \int_{x=0}^{1}\int_{y=0}^{1} \frac{x}{1+xy}\, dy\, dx \\
&= \int_{x=0}^{1} [\ln(1+xy)]_{y=0}^{1}\, dx \\
&= \int_{x=0}^{1} \ln(1+x)\, dx \\
&= \int_{1}^{2} \ln(u)\, du \\
&= [u\ln(u) - u]_1^2 \\
&= 2\ln(2) - 1.
\end{aligned}$$

Problem 3: Evaluate the integral

$$\iint_{\mathcal{D}} xy\, dA$$

where $\mathcal{D}$ is the region bounded by the line $y = x-1$ and the parabola $y^2 = 2x+6$.

The domain $\mathcal{D}$ is *horizontally simple* (Case (B) in the text) so it makes things easier to rotate the picture:



Then we write the iterated integral:

$$\begin{aligned}
\iint_{\mathcal{D}} xy\, dA &= \int_{y=-2}^{4} \int_{x=\frac{y^2}{2}-3}^{y+1} xy\, dx\, dy \\
&= \int_{y=-2}^{4} y \left[\frac{x^2}{2}\right]_{x=\frac{y^2}{2}-3}^{y+1} dy \\
&= \frac{1}{2}\int_{-2}^{4} y\left((y+1)^2 - \left(\frac{y^2}{2}-3\right)^2\right) dy \\
&= \frac{1}{2}\int_{-2}^{4} \left(-\frac{y^5}{4} + 4y^3 + 2y^2 - 8y\right) dy \\
&= 36.
\end{aligned}$$