Math 13 Spring 2017
Final Examination
Elements of solution

(1) **For each of the following assertions, select the correct ending.**

(a) **If** $f(x, y)$ **is a function, then** $\int_0^2 \int_0^x f(x, y)\, dy\, dx = \ldots$

☐ $\int_0^2 \int_0^2 f(x, y)\, dx\, dy$.

☐ $\int_0^2 \int_0^x f(x, y)\, dx\, dy$.

☐ $\int_0^2 \int_0^y f(x, y)\, dx\, dy$.

☒ $\int_0^2 \int_y^2 f(x, y)\, dx\, dy$.

☐ $\int_0^x \int_0^2 f(x, y)\, dx\, dy$.

☐ none of the above.

(b) **A potential for the vector field** $\mathbf{F}(x, y, z) = \sin(yz^2)\mathbf{i} + xz^2 \cos(yz^2)\mathbf{j} + 2xyz \cos(yz^2)\mathbf{k}$ **is...**

☐ $\langle 0\,,\ -xz^4 \sin(yz^2)\,,\ -4xy^2z^2 \sin(yz^2)\rangle$.

☐ $\langle \sin(yz^2)\,,\ xz^2 \cos(yz^2)\,,\ 2xyz \cos(yz^2)\rangle$.

☒ $x \sin(yz^2) + 2$.

☐ $-xz^2(z^2 + 4y^2) \sin(yz^2)$.

☐ none of the above.

(c) **Let** $\mathbf{F}(x, y, z)$ **be a vector field defined on an open, connected and simply connected domain. Then F is conservative if and only if...**

☐ $\mathbf{F} = \nabla f$ for some scalar function $f(x, y, z)$.

☐ $\nabla \times \mathbf{F} = \mathbf{0}$.

☐ $\oint_\Gamma \mathbf{F} \cdot d\mathbf{r} = 0$ for any closed curve Γ in the domain of $\mathbf{F}$.

☒ all of the above.

☐ none of the above.

(d) **The Fundamental Theorem of line integrals...**

☐ applies to scalar functions.

☒ only applies to conservative vector fields.

☐ applies to all vector fields.

☐ only applies to plane curves, not to curves in 3-dimensional space.

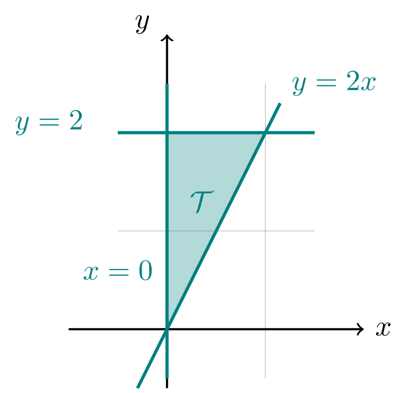
☐ none of the above.

(2) **Calculate** $\iint_{\mathcal{R}} \frac{x}{1+xy}\, dA$ **where** $\mathcal{R} = [0,1] \times [0,1]$**.**

$$\begin{aligned}
\iint_{\mathcal{R}} \frac{x}{1+xy}\, dA &= \int_{x=0}^{1} \int_{y=0}^{1} \frac{x}{1+xy}\, dy\, dx \\
&= \int_{x=0}^{1} [\ln(1+xy)]_{y=0}^{1}\, dx \\
&= \int_{x=0}^{1} \ln(1+x)\, dx \\
&= \int_{1}^{2} \ln(u)\, du \\
&= [u\ln(u) - u]_1^2 = \boxed{2\ln(2) - 1}.
\end{aligned}$$

(3) **Find the volume of the solid bounded by the planes** $x = 0$**,** $y = 2$**,** $z = 0$**,** $y = 2x$ **and the surface** $z = y^2$**.**

The solid consists of the part of the solid cylinder with base $\mathcal{T}$ pictured below and comprised between the xy-plane and the surface $z = y^2$.

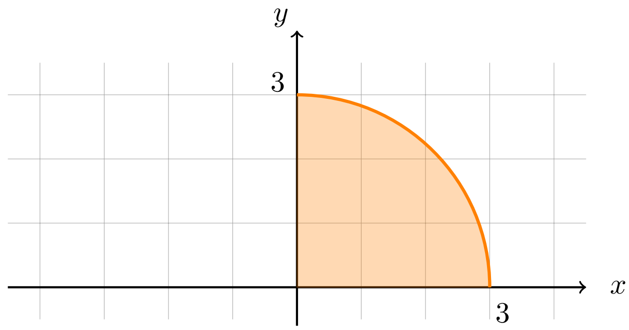


so the volume is equal to

$$\iint_{\mathcal{T}} y^2\, dA = \int_0^1 \int_{2x}^{2} y^2\, dy\, dx = \frac{8}{3} \int_0^1 (1 - x^3)\, dx = \boxed{2}.$$

(4) **Consider the iterated integral** $\mathcal{I} = \int_0^3 \int_0^{\sqrt{9-x^2}} (x^2+y^2)\,dy\,dx$**.**

(a) **Sketch the domain of integration in the** xy**-plane.**



(b) **Evaluate** $\mathcal{I}$**.**

Using polar coordinates, $\mathcal{I} = \int_0^{\frac{\pi}{2}} \int_0^3 r^2 \cdot r\,dr\,d\theta = \boxed{\frac{81\pi}{8}}$.

(5) **Let** $\mathcal{W}$ **be a solid with density** $\delta(x,y,z)$**. Match the following quantities with their expression:**

A. Total mass of $\mathcal{W}$

B. y**-coordinate of the center of mass**

C. Mean value of the density

(blank)	$\dfrac{\iiint_{\mathcal{W}} x\,dV}{\iiint_{\mathcal{W}} dV}$
(blank)	$\dfrac{\iiint_{\mathcal{W}} y\,dV}{\iiint_{\mathcal{W}} dV}$
(blank)	$\dfrac{\iiint_{\mathcal{W}} z\,dV}{\iiint_{\mathcal{W}} dV}$
(blank)	$\iiint_{\mathcal{W}} 1\,dV$
A	$\iiint_{\mathcal{W}} \delta(x,y,z)\,dV$
C	$\dfrac{\iiint_{\mathcal{W}} \delta(x,y,z)\,dV}{\iiint_{\mathcal{W}} dV}$
(blank)	$\dfrac{\iiint_{\mathcal{W}} x\,\delta(x,y,z)\,dV}{\iiint_{\mathcal{W}} \delta(x,y,z)\,dV}$
B	$\dfrac{\iiint_{\mathcal{W}} y\,\delta(x,y,z)\,dV}{\iiint_{\mathcal{W}} \delta(x,y,z)\,dV}$
(blank)	$\dfrac{\iiint_{\mathcal{W}} z\,\delta(x,y,z)\,dV}{\iiint_{\mathcal{W}} \delta(x,y,z)\,dV}$

(6) **Let $\mathcal{D}$ be the domain bounded by the curves $y = x$, $y = 3x$, $xy = 1$ and $xy = 3$ in the first quadrant. What does the integral $\iint_{\mathcal{D}} xy\,dA$ become under the change of variables $x = \frac{u}{v}$, $y = v$?**

- ☐ $\int_1^3 \int_1^3 u\,dv\,du.$
- ☐ $\int_1^3 \int_{\sqrt{u}}^{\sqrt{3u}} u\,dv\,du.$
- ☐ $\int_1^3 \int_{u^2}^{3u^2} u\,dv\,du.$
- ☐ $\int_1^3 \int_1^3 \frac{u}{v}\,dv\,du.$
- ☒ $\int_1^3 \int_{\sqrt{u}}^{\sqrt{3u}} \frac{u}{v}\,dv\,du.$
- ☐ $\int_1^3 \int_{u^2}^{3u^2} \frac{u}{v}\,dv\,du.$

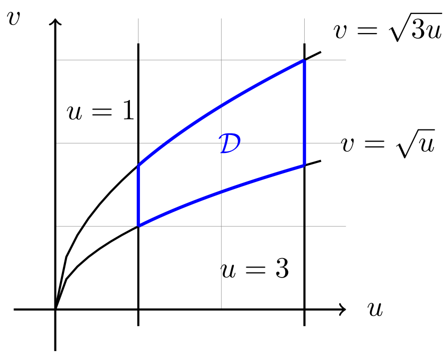
Under the change of variables, the boundary curves

$$y = x \quad , \quad y = 3x \quad , \quad xy = 1 \quad \text{and} \quad xy = 3$$

respectively become

$$v^2 = u \quad , \quad v^2 = 3u \quad , \quad u = 1 \quad \text{and} \quad u = 3$$

giving the domain of integration $\mathcal{D}$ below.



The Jacobian of the transformation is $\begin{vmatrix} \frac{1}{v} & -\frac{u}{v^2} \\ 0 & 1 \end{vmatrix} = \frac{1}{v}.$

(7) **Let Γ be the curve parametrized by $\mathbf{r}(t) = t\mathbf{i} + 2t\mathbf{j} + e^t\mathbf{k}$ for $0 \leq t \leq 1$.**

(a) **Determine the starting point and the end point of Γ.**

Setting $t = 0$ gives $P = (0, 0, 1)$ and $t = 1$ leads to $Q = (1, 2, e)$.

(b) **Consider the function $f(x, y, z) = x\sqrt{y}\ln(1 + x - y + z)$ and let $\mathbf{F} = \nabla f$. Evaluate $\int_\Gamma \mathbf{F} \cdot d\mathbf{r}$.**

By the Fundamental Theorem for line integrals, $\int_\Gamma \nabla f \cdot d\mathbf{r} = f(Q) - f(P) = \boxed{\sqrt{2}}.$

(8) **Let Σ be the part of the sphere with equation $x^2 + y^2 + z^2 = 1$ in the $x < 0$ region. The integral $\iint_\Sigma x\, dS$ is equal to...**

- ☐ $\displaystyle\int_0^\pi \int_0^\pi \cos\theta \sin\varphi \, d\theta \, d\varphi.$
- ☐ $\displaystyle\int_0^\pi \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \cos\theta \sin\varphi \, d\theta \, d\varphi.$
- ☐ $\displaystyle\int_0^\pi \int_0^{2\pi} \cos\theta \sin\varphi \, d\theta \, d\varphi.$
- ☐ $\displaystyle\int_0^\pi \int_0^{2\pi} \cos\theta \sin^2\varphi \, d\theta \, d\varphi.$
- ☒ $\displaystyle\int_0^\pi \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \cos\theta \sin^2\varphi \, d\theta \, d\varphi.$
- ☐ $\displaystyle\int_0^\pi \int_0^\pi \cos\theta \sin^2\varphi \, d\theta \, d\varphi.$
- ☐ $\displaystyle\int_0^\pi \int_0^{2\pi} \int_0^1 \cos\theta \sin^2\varphi \, \rho \, d\rho \, d\theta \, d\varphi.$
- ☐ $\displaystyle\int_0^\pi \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \int_0^1 \cos\theta \sin^2\varphi \, \rho \, d\rho \, d\theta \, d\varphi.$
- ☐ $\displaystyle\int_0^\pi \int_0^\pi \int_0^1 \cos\theta \sin^2\varphi \, \rho \, d\rho \, d\theta \, d\varphi.$

Using spherical coordinates, we get

$$x = \cos\theta \sin\varphi$$

and the magnitude of the normal vector is

$$\|N(\theta, \varphi)\| = \sin\varphi$$

so the integrand must be

$$\cos\theta \sin^2\varphi.$$

As for the range of the parameters, there is no restriction on z so

$$0 \leq \varphi \leq \pi$$

and the condition $x < 0$ leads to

$$\frac{\pi}{2} \leq \theta \leq \frac{3\pi}{2}.$$

(9) **Let Σ be the surface parametrized by $S(u, v) = \big(u^2\,,\, v^2\,,\, u + 2v\big)$ for $0 \leq u \leq 2$ and $0 \leq v \leq 2$. Find an equation for the plane tangent to Σ at $(1, 1, 3)$.**

Note that $(1, 1, 3) = S(1, 1)$ so a normal vector $\mathbf{N}$ for the tangent plane can be obtained by taking the crossed product $\mathbf{T}_u(1, 1) \times \mathbf{T}_v(1, 1)$ where

$$\mathbf{T}_u(u, v) = \langle 2u\,,\, 0\,,\, 1\rangle \qquad \text{and} \qquad \mathbf{T}_v(u, v) = \langle 0\,,\, 2v\,,\, 2\rangle\,.$$

Therefore, we choose

$$\mathbf{N} = \langle 2\,,\, 0\,,\, 1\rangle \times \langle 0\,,\, 2\,,\, 2\rangle = \langle -2\,,\, -4\,,\, 4\rangle = -2 \cdot \langle 1\,,\, 2\,,\, -2\rangle$$

and the equation is of the form $x + 2y - 2z + d = 0$. Finally we determine d so that $(1, 1, 3)$ satisfies the equation:

$$\boxed{x + 2y - 2z + 3 = 0}.$$

(10) **Evaluate** $\displaystyle\int_\Sigma \mathbf{F}\cdot d\mathbf{S}$ **where**

$$\mathbf{F}(x,y,z) = \left\langle xy\,,\, 4x^2, yz\right\rangle$$

and Σ **is the surface** $z = xe^y$ **for** $0 \le x \le 1$ **and** $0 \le y \le 1$**, oriented in the positive** x**-direction.**

A parametrization for Σ is

$$S(u,v) = (u, v, ue^v) \qquad , \qquad 0 \le u \le 1 \quad , \quad 0 \le v \le 1$$

with tangent vectors $\mathbf{T}_u = \langle 1\,,\, 0\,,\, e^v\rangle$ and $\mathbf{T}_v = \langle 0\,,\, 1\,,\, ue^v\rangle$ so that

$$\mathbf{T}_u \times \mathbf{T}_v = \langle -e^v\,,\, -ue^v\,,\, 1\rangle\,.$$

Given the orientation on Σ, we choose $\mathbf{N}(u,v) = -\mathbf{T}_u \times \mathbf{T}_v$ and since

$$\mathbf{F}(S(u,v)) = \left\langle uv\,,\, 4u^2\,,\, uve^v\right\rangle$$

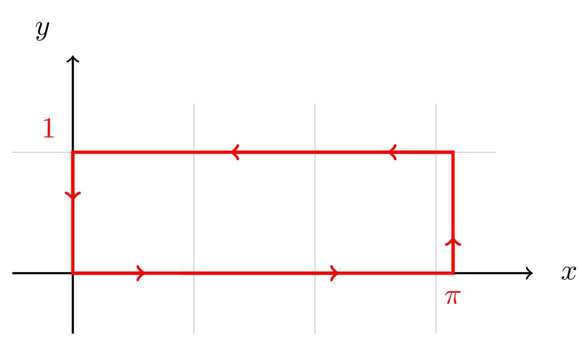
we get

$$\begin{aligned}\int_\Sigma \mathbf{F}\cdot d\mathbf{S} &= \int_0^1\int_0^1 \left\langle uv\,,\, 4u^2\,,\, uve^v\right\rangle \cdot \langle e^v\,,\, ue^v\,,\, -1\rangle\, du\, dv \\ &= \int_0^1\int_0^1 4u^3e^v\, du\, dv \\ &= 4\int_0^1 u^3\, du \int_0^1 e^v\, dv = \boxed{e-1}.\end{aligned}$$

(11) **Calculate the circulation of the vector field**

$$\mathbf{F}(x,y) = \left\langle y^3 + 2xy + 2y - e^{-x^2}\,,\, x^2 + 3xy^2 + 5x + \cos(\sqrt{y})\right\rangle$$

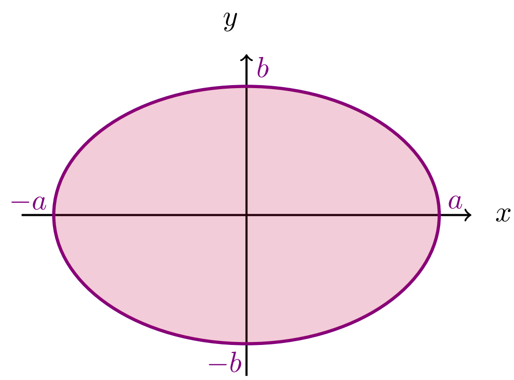
along the closed rectangular path Γ **drawn below.**



By Green's Theorem,

$$\begin{aligned}\oint_\Gamma \mathbf{F}\cdot d\mathbf{r} &= \iint_{[0,\pi]\times[0,1]} \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dA \\ &= \int_0^1\int_0^\pi (2x + 3y^2 + 5 - (3y^2 + 2x + 2))\, dx\, dy \\ &= 3\int_0^1\int_0^\pi dx\, dy = \boxed{3\pi}.\end{aligned}$$

(12) **Use Green's Theorem to calculate the surface of the elliptic domain $\mathcal{E}$ with equation $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} \leq 1$.**



Parameterizing the ellipse by $\begin{cases} x(t) = a\cos t \\ y(t) = b\sin t \end{cases}$ for $0 \leq t \leq 2\pi$, it is a consequence of Green's Theorem that

$$\begin{aligned}\text{Area}(\mathcal{E}) &= \frac{1}{2}\oint_{\partial\mathcal{E}} x\,dy - y\,dx \\ &= \frac{1}{2}\int_0^{2\pi} (a\cos t \cdot b\cos t - b\sin t \cdot (-a\sin t))\,dt \\ &= \frac{ab}{2}\int_0^{2\pi} dt = \boxed{\pi ab}.\end{aligned}$$

(13) **Let $\mathbb{S}$ be the hemisphere $x^2 + y^2 + z^2 = 9$ in the $z \geq 0$ region, oriented upward and $\mathbf{F}(x, y, z) = 2x\cos z\mathbf{i} + e^x \sin z\mathbf{j} + xe^y\mathbf{k}$. Evaluate $\iint_{\mathbb{S}} \operatorname{curl}\mathbf{F} \cdot d\mathbf{S}$.**

By Stokes' Theorem, $\iint_{\mathbb{S}} \operatorname{curl}\mathbf{F} \cdot d\mathbf{S} = \oint_{\Gamma} \mathbf{F} \cdot d\mathbf{r}$ where Γ is the circle parametrized by

$$\mathbf{r}(t) = \langle 3\cos t\,,\ 3\sin t\,,\ 0\rangle \quad , \quad 0 \leq t \leq 2\pi.$$

Since

$$\mathbf{F}(\mathbf{r}(t)) = \left\langle 6\cos t\,,\ 0\,,\ \cos t e^{\sin t}\right\rangle \qquad \text{and} \qquad \mathbf{r}'(t) = \langle -3\sin t\,,\ 3\cos t\,,\ 0\rangle\,,$$

we get

$$\iint_{\mathbb{S}} \operatorname{curl}\mathbf{F} \cdot d\mathbf{S} = \boxed{-18\int_0^{2\pi} \cos t \sin t\,dt = 0}.$$

(14) **Let Σ denote the part of the paraboloid $y = x^2 + z^2$ with $0 \le y \le 1$. Consider**

$$\mathbf{F}(x,y,z) = \left\langle z\sqrt{7} + y + e^{x^2}\,,\ xy + \sin\sqrt{y}\,,\ 0 \right\rangle \quad \textbf{and} \quad \mathbf{G} = \operatorname{curl}\mathbf{F}.$$

Evaluate $\iint_\Sigma \mathbf{G} \cdot d\mathbf{S}$, using normal vectors pointing in the positive y-direction.

By Stokes' Theorem, the flux of $\operatorname{curl}\mathbf{F}$ is the same across any surface having the same compatibly oriented boundary as Σ. This boundary is the circle centered at $(0,1,0)$ with radius 1. It is also the boundary of the disk $\mathcal{D}$ centered at $(0,1,0)$ with radius 1 in the $y = 1$ plane. A parametrization for $\mathcal{D}$ is

$$S(r,\theta) = (r\cos\theta\,,\ 1\,,\ r\sin\theta) \qquad , \qquad 0 \le r \le 1 \quad , \quad 0 \le \theta \le 2\pi$$

with normal vector $\langle 0\,,\ r\,,\ 0\rangle$ compatible with the orientation of the circle inherited from that of Σ. Since $\operatorname{curl}\mathbf{F}(x,y,z) = \left\langle 0\,,\ \sqrt{7}\,,\ y-1\right\rangle$, we get

$$\mathbf{G}(S(r,\theta)) = \left\langle 0\,,\ \sqrt{7}\,,\ 0\right\rangle$$

and

$$\begin{aligned}
\iint_\Sigma \mathbf{G} \cdot d\mathbf{S} &= \iint_{\mathcal{D}} \mathbf{G} \cdot d\mathbf{S} \\
&= \int_0^{2\pi}\int_0^1 \left\langle 0\,,\ \sqrt{7}\,,\ 0\right\rangle \cdot \langle 0\,,\ r\,,\ 0\rangle \, dr\, d\theta \\
&= 2\pi\sqrt{7}\int_0^1 r\, dr \\
&= \boxed{\pi\sqrt{7}}.
\end{aligned}$$

(15) **Let $\mathbf{F}$ be a vector field defined on a simple solid region $\mathcal{W}$ with boundary the closed surface $\mathcal{S}$. Then, $\iint_{\mathcal{S}} \operatorname{curl}\mathbf{F} \cdot d\mathbf{S}$ is...**

- ☐ a vector.
- ☐ $\iiint_{\mathcal{W}} \operatorname{div}\mathbf{F}\, dV$.
- ☒ 0 be the Divergence Theorem.
- ☒ 0 by Stokes Theorem.
- ☐ not well-defined.
- ☐ none of the above.

One can apply Stokes' Theorem and notice that ∂S is empty so

$$\iint_{\mathcal{S}} \operatorname{curl}\mathbf{F} \cdot d\mathbf{S} = \oint_{\emptyset} \mathbf{F} \cdot d\mathbf{r} = 0$$

or the Divergence Theorem, leading to

$$\iint_{\mathcal{S}} \operatorname{curl}\mathbf{F} \cdot d\mathbf{S} = \iiint_{\mathcal{W}} \underbrace{\operatorname{div}(\operatorname{curl}\mathbf{F})}_{=0} = 0.$$