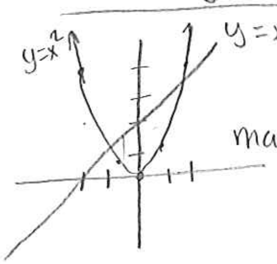
Math 13 Worksheet #4: Applications of double integration

(1) Find the mass and center of mass of the lamina that occupies the region D and has the density function $\rho(x, y) = kx$, where D is bounded by $y = x^2$ and $y = x + 2$.

DRAW Region



Find intersections

$x^2 = x+2 \Rightarrow x^2 - x + 2 = 0 \Rightarrow (x-2)(x+1) = 0 \Rightarrow$ intersect at $x = 2, \& x = -1$

$$\text{mass} = M = \iint_D \rho(x,y)\,dA = \int_{-1}^{2}\int_{x^2}^{x+2} kx\,dy\,dx = \int_{-1}^{2} kxy\Big|_{x^2}^{x+2} dx$$

$$\int_{-1}^{2} kx(x+2-x^2)\,dx = \int_{-1}^{1} k(x^2+2x-x^3)\,dx = k\left[\frac{x^3}{3} + x^2 - \frac{x^4}{4}\Big|_{-1}^{2}\right]$$

$$= k\left[\frac{8}{3} + 4 - \frac{16}{4} - \left(\frac{-1}{3} + 1 - \frac{1}{4}\right)\right] = \frac{9}{4}$$

see second page for details

$$M_y = \iint_D x\rho(x,y)\,dA = \int_{-1}^{2}\int_{x^2}^{x+2} kx^2\,dy\,dx = \frac{63}{20}$$

$$M_x = \iint_D y\rho(x,y)\,dA = \int_{-1}^{2}\int_{x^2}^{x+2} kxy\,dy\,dx = \frac{45}{4}$$

Center of mass

$$(\bar{x}, \bar{y}) = \left(\frac{M_y}{M}, \frac{M_x}{M}\right) =$$

(2) A lamina occupies the region inside the circle $x^2 + y^2 = 2y$ and outside the circle $x^2 + y^2 = 1$. Find the center of mass if the density is inversely proportional to its distance. FROM the origin

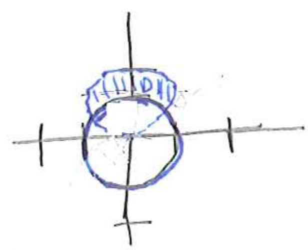
Circle 1: $x^2 + y^2 = 2y$

$x^2 + y^2 - 2y = 0$

$x^2 + (y-1)^2 = 1$

Circle 2: $x^2 + y^2 = 1$

DRAW



— Circle 2

----- circle 1

DENSITY: $\rho(x,y) = \dfrac{k}{\sqrt{x^2+y^2}} = \dfrac{k}{r}$

DEFINE REGION D: i.e. find intersections.

Circle 2: $x^2 = 1 - y^2$ Plug into circle 1.

$1 - y^2 + y^2 - 2y = 0 \Rightarrow 2y = 1 \Rightarrow y = 1/2$

$\Rightarrow x = \pm\sqrt{1 - 1/4} = \pm\frac{\sqrt{3}}{2}$

Integration in Cartesian coordinates is Messy. lets use polar. see page 3.

(1) $M_y = \int_{-1}^{2} k\, x^2 y \Big|_{x^2}^{x+2} dy = \int_{-1}^{2} [k x^2 (x+2-x^2)]\, dx$

$= k\int_{-1}^{2} [x^3 + 2x^2 - x^4]\, dx = k\left[\frac{x^4}{4} + \frac{2x^3}{3} - \frac{x^5}{5}\Big|_{-1}^{2}\right]$

$= k\left[4 + \frac{16}{3} - \frac{32}{5} - \left(\frac{1}{4} - \frac{2}{3} + \frac{1}{5}\right)\right] = \frac{63}{20}$

$M_x = \int_{-1}^{2} \frac{kx}{2} y^2 \Big|_{x^2}^{x+2} dx = \frac{k}{2}\int_{-1}^{2} x[(x+2)^2 - x^4]\, dx$

$= \frac{k}{2}\int_{-1}^{2} x[x^2+4x+4-x^4]\, dx = \frac{k}{2}\int_{-1}^{2} (x^3+4x^2+4x-x^5)\, dx$

$= \frac{k}{2}\left[\frac{x^4}{4} + \frac{4x^3}{3} + 2x^2 - \frac{x^6}{6}\Big|_{-1}^{2}\right] = \frac{k}{2}\left[4 + \frac{32}{3} + 8 + \frac{64}{6} - \left(\frac{1}{4} - \frac{4}{3} + 2 - \frac{1}{6}\right)\right]$

$= \frac{45}{4}$

Extra

Inertia

$I_y = \iint_D x^2 \rho(x,y)\, dA = \int_{-1}^{2}\int_{x^2}^{x+2} x^2(kx)\, dy\, dx =$

$I_x = \iint_D y^2 \rho(x,y)\, dA = \int_{-1}^{2}\int_{x^2}^{x+2} y^2(kx)\, dy\, dx =$

$I_0 = \iint_D (x^2+y^2)\rho(x,y)\, dA = \iint_D x^2 \rho(x,y)\, dA + \iint_D y^2 \rho(x,y)\, dA$

$= I_y + I_x.$ =

Polar coordinates $\quad x = r\cos\theta \quad y = r\sin\theta$

Circle 1: $x^2 + y^2 - 2y = 0 \Rightarrow r^2 - 2r\sin\theta = 0 \Rightarrow r = 2\sin\theta$

Circle 2: $r = 1$

$$y = 1/2 \Rightarrow \theta = \pi/6, \ \frac{5\pi}{6}$$

$$\Rightarrow \quad 1 \le r \le 2\sin\theta, \quad \pi/6 \le \theta \le \frac{5\pi}{6}$$

$$M = \iint_D \rho(x,y)\,dA = k\int_{\frac{\pi}{6}}^{5\pi/6}\int_1^{2\sin\theta} \frac{1}{r}\, r\,dr\,d\theta = k\int_{\pi/6}^{\frac{5\pi}{6}} (2\sin\theta - 1)\,d\theta$$

$$= k\left(-2\cos\theta\Big|_{\pi/6}^{5\pi/6} - \theta\Big|_{\pi/6}^{5\pi/6}\right) =$$

$$= k\left(-2\left(\frac{-\sqrt{3}}{2} - \frac{\sqrt{3}}{2}\right) - \left(\frac{5\pi}{6} - \pi/6\right)\right) = k\left(2\sqrt{3} - \frac{4\pi}{6}\right)$$

$$M_y = k\iint_D x\rho(x,y)\,dA = k\int_{\pi/6}^{\frac{5\pi}{6}}\int_1^{2\sin\theta} \frac{r}{r}\cos\theta\,dr\,d\theta$$

$$= k\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} \cos\theta(2\sin\theta - 1)\,d\theta$$

$$= k\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} (\sin(2\theta) - \cos(\theta))\,d\theta$$

$$= k\left[-\frac{1}{2}\cos(2\theta) + \sin(\theta)\Big|_{\pi/6}^{5\pi/6}\right]$$

$$= k\left[-\frac{1}{2}\left(\cos(5\pi/3) - \cos(\pi/3)\right) + \underbrace{(1/2 - 1/2)}_{0}\right]$$

$$= k\left[-\frac{1}{2}\underbrace{(1/2 - 1/2)}_{0}\right] = 0$$

$$M_x = \iint_D y\,\rho(x,y)\,dA = k\int_{\pi/6}^{\frac{5\pi}{6}} \int_1^{2\sin\theta} \frac{r\sin\theta}{r}\,dr\,d\theta$$

$$= k\int_{\pi/6}^{\frac{5\pi}{6}} \sin\theta\,(2\sin\theta - 1)\,d\theta = k\int_{\pi/6}^{\frac{5\pi}{6}} (2\sin^2\theta - \sin\theta)\,d\theta$$

$$= k\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} (1-\cos 2\theta - \sin\theta)\,d\theta = k\left(\theta - \frac{\sin 2\theta}{2} + \cos\theta\right)\Bigg|_{\pi/6}^{5\pi/6}$$

$$= k\left[\frac{5\pi}{6} - \frac{1}{2}\sin\left(\frac{5\pi}{3}\right) + \cos(5\pi/6) - \left(\frac{\pi}{6} - \frac{1}{2}\sin(\pi/3) + \cos(\pi/6)\right)\right]$$

$$= k\left[\frac{2\pi}{3} + \frac{1}{2}\left(\frac{\sqrt{3}}{2}\right) + \frac{-\sqrt{3}}{2} + \frac{1}{2}\left(\frac{\sqrt{3}}{2}\right) - \frac{\sqrt{3}}{2}\right] = k\left[\frac{2\pi}{3} + \frac{\sqrt{3}}{2} - \sqrt{3}\right]$$

$$= k\left[\frac{2\pi}{3} + \frac{\sqrt{3}}{2}\right.$$