Math 13 Worksheet #19: Stokes' Theorem

(1) Verify Stokes' Theorem for the vector field $\boldsymbol{F}(x,y,z) = < -y, x, e^z >$ on the surface defined by $S = \{(x,y,z) : z = 1 - x^2 - y^2,\ x^2 + y^2 \le 1\}$, with outward unit normal vector.

C: is defined by $\bar{r}(\theta) = < \cos\theta, \sin\theta, 0 > \quad 0 \le \theta \le 2\pi.$

$\int_C \bar{F} \cdot d\bar{r} = \int_0^{2\pi} < -\sin\theta, \cos\theta, 1 > \cdot < -\sin\theta, \cos\theta, 0 > d\theta \quad \left(\int_C \bar{F}\cdot d\bar{r} = \int_0^{2\pi} F \cdot r' \, d\theta\right)$

$= \int_0^{2\pi} \sin^2\theta + \cos^2\theta + 0 \, d\theta = 2\pi.$

Now the other way

$\text{curl}\,\bar{F} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \partial x & \partial y & \partial z \\ -y & x & e^z \end{vmatrix} = < 0, 0, 1+1 > = < 0, 0, 2 >$

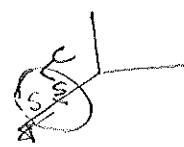
let $G = z - g(x,y) = 0 = z - (1 - x^2 - y^2) \quad \bar{N} = \nabla G = < 2x, 2y, 1 >$

$\iint_S \text{curl}\,F \cdot \bar{N} \, dS = \iint_D < 0,0,2 > \cdot < 2x, 2y, 1 > dA = 2 \iint_D dA = 2 \int_0^1 \int_0^{2\pi} r \, d\theta \, dr$

$= 4\pi \int_0^1 r \, dr = 2\pi \ r^2 \Big|_0^1 = 2\pi$

The two integrals match so we have verified Stokes' Thm.

(2) Use Stokes' Theorem to evaluate to evaluate the integral of the vector field $\boldsymbol{F}(x,y,z) = < e^{xyz}, -xy^2z, xyz^2 >$ around the curve C given by $z^2 + y^2 = 9$ in the plane $x = 5$ and transversed in the counterclockwise direction when viewed from the right (i.e. where $x > 5$.)



$\int_C \bar{F} \cdot d\bar{r} = \iint_S \text{curl}\,\bar{F} \cdot \bar{n} \, dS$

$\text{curl}\,\bar{F} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \partial x & \partial y & \partial z \\ e^{xyz} & -xy^2z & xyz^2 \end{vmatrix} = < xz^2 + xy^2, -(yz^2 - xye^{xyz}), -y^2z - xze^{xyz} >$

S: is defined by $\bar{r}(r,\theta) = < 5, r\cos\theta, r\sin\theta > \quad \begin{cases} 0 \le r \le 3 \\ 0 \le \theta \le 2\pi \end{cases}$

$\bar{r}_r = < 0, \cos\theta, \sin\theta > \quad \bar{r}_\theta = < 0, -r\sin\theta, r\cos\theta >$

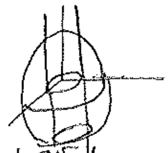
$\bar{r}_r \times \bar{r}_\theta = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ 0 & \cos\theta & \sin\theta \\ 0 & -r\sin\theta & r\cos\theta \end{vmatrix} = < r, 0, 0 >$

$\iint_S \text{curl}\,\bar{F} \cdot \bar{n} \, dS = \int_0^{2\pi} \int_0^3 < 5z^2 + 5y^2, -(yz^2 - 5ye^{5yz}), -y^2z - 5ze^{5yz} > \cdot < r, 0, 0 > r\,dr\,d\theta$

$= \int_0^{2\pi} \int_0^3 5r^3 \, dr \, d\theta = 10\pi \frac{r^4}{4} \Big|_0^3 = \frac{5}{2}\pi (3^4)$

(3) Evaluate $\iint\limits_S \mathrm{curl}\boldsymbol{F} \cdot \boldsymbol{n} dS$, where S is the cap of the unit sphere that lies below the xy−plane and inside the cylinder $x^2+y^2=\frac{1}{9}$ with outwards-pointing normal vector and where $\boldsymbol{F}(x,y,z) =< -yz^2, xz^2, 3^{-xyz} >$.

$\iint\limits_S \mathrm{curl}\bar{F}\cdot\bar{n}\, dS = \int_C \bar{F}\cdot d\bar{r}$



Surface on sphere: $x^2+y^2+z^2=1 \Rightarrow \quad z^2 = 1-1/9 = 8/9 \Rightarrow z = \pm 2\sqrt{2}/3$ since on bottom $z=-2\sqrt{2}/3$.

C: $x^2+y^2=1/9 \quad z=-2\sqrt{2}/3$

Parameterized: $\bar{r}(\theta) = < 1/3\cos\theta, 1/3\sin\theta, -2\sqrt{2}/3 > \quad 0 \le \theta \le 2\pi$

$\int_C \bar{F}\cdot d\bar{r} = \int_0^{2\pi} F\cdot r'(\theta)\, d\theta = \int_0^{2\pi} <-8/9(1/3)\sin\theta, 8/9(1/3)\cos\theta, 3^{xyz}> \cdot <-1/3\sin\theta, 1/3\cos\theta, 0>\, d\theta$

$= \int_0^{2\pi} [8/81 \sin^2\theta + 8/81\cos^2\theta + 0]\, d\theta = 8/81(2\pi)$

(4) For each of the following problems explain why Stokes' Theorem does not apply.

(a) S is the pyramid with vertices at $(0,0,6)$, $(2,0,0)$, $(-2,0,0)$, $(0,3,0)$, and $(0,-3,0)$.

The surface is not piecewise smooth.

(b) $\boldsymbol{F}(x,y,z) =< \ln(xy+1) + 5^x 3^y 2^z, 4xz^2 >$, and C is the boundary of the square in the plane $z=6$ and with vertices $(2,0,6)$, $(-2,0,6)$, $(2,4,6)$, and $(-2,4,6)$.

$\bar{F}$ does not have continuous partial derivatives for each of the components for any pt along the surface where $x=0$.