Math 13 Worksheet #18: Divergence Thm

(1) Verify the conclusion of the Divergence Theorem for the vector field $F(x,y,z) = < x^2, y^2, z^2 >$ with the region R the unit ball centered at the origin.

1st $\iint_{S} \bar{F}\cdot\bar{n}\, ds$

surface S: $\bar{r}(\phi,\theta) = < \cos\theta \sin\phi, \sin\theta \sin\phi, \cos\phi >$ $\quad 0 \le \theta \le 2\pi,\ 0 \le \phi \le \pi.$

$\bar{r}_\phi \times \bar{r}_\theta = < \cos\theta \sin^2\phi, \sin\theta \sin^2\phi, \cos\phi \sin\phi > \Rightarrow n = \bar{r}(\phi,\theta)$

$F = < \cos^2\theta \sin^2\phi, \sin^2\theta \sin^2\phi, \cos^2\phi >.$

on S

$$\iint_S \bar{F}\cdot\bar{n} = \int_0^{2\pi}\int_0^{\pi} (\cos^3\theta \sin^3\phi + \sin^3\theta \sin^3\phi + \cos^3\phi)\, d\phi\, d\theta$$

$$= \int_0^{2\pi} [\cos^3\theta + \sin^3\theta] \left(-\tfrac{1}{3}\sin^2\phi \cos\phi - \tfrac{2}{3}\cos\phi\right) - \tfrac{1}{3}\cos^2\phi \sin\phi - \tfrac{2}{3}\phi \Big]_0^{\pi} d\theta$$

$$= \int_0^{2\pi} \frac{4}{3}(\cos^3\theta + \sin^3\theta)\, d\theta = \frac{4}{3}\left(\sin\theta - \tfrac{1}{3}\sin^3\theta - \cos\theta + \tfrac{1}{3}\cos^3\theta\right)\Big|_0^{2\pi}$$

$$= 0$$

Now

$$\iiint_R \operatorname{div} F\, dV = 2\int_0^{2\pi}\int_0^{\pi}\int_0^1 [\rho \cos\theta \sin\phi + \rho \sin\theta \sin\phi + \rho\cos\phi]\, \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta =$$

see Page 3.

(2) Evaluate the integral $\iint_S F(x,y,z)\cdot n\, dS$ for $F(x,y,z) = < xz, yz, xyz >$ and S is the surface of the cylinder with equation $x^2 + y^2 = 9$ for $-2 \le z \le 2$ and n being the outward pointing normal vector.

$$\iint_S \bar{F}\cdot\bar{n}\, ds = \iiint_V \operatorname{div} F\, dV$$

$$\operatorname{div} F = z + z + xy = 2z + xy$$

$$\iint_S \bar{F}\cdot\bar{n}\, ds = \iint_D \int_{-2}^{2} 2z + xy\, dz\, dA \qquad \text{where } D = \{(x,y) : x^2+y^2 \le 9\}$$

$$= \iint_D z^2 + xyz \Big|_{-2}^{2} dA = \iint_D 4 - 4 + xy(2+2)\, dA$$

$$= \iint_D 4xy\, dA \qquad \text{let } \begin{matrix} x = r\cos\theta \\ y = r\sin\theta \end{matrix} \qquad \begin{matrix} 0 \le r \le 3 \\ 0 \le \theta \le 2\pi \end{matrix}$$

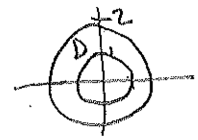
$$= \int_0^{2\pi}\int_0^3 r^2 \cos\theta \sin\theta\, r\, dr\, d\theta = \int_0^{2\pi} \frac{r^4}{4}\Big|_0^3 \cos\theta \sin\theta\, d\theta$$

$$= \frac{3^4}{4}\ \frac{1}{2}\sin^2\theta \Big|_0^{2\pi} = 0.$$

(3) Evaluate the integral $\iint_S \boldsymbol{F}(x,y,z) \cdot \boldsymbol{n} dS$ for $\boldsymbol{F}(x,y,z) = < \sin y \cos z, yz^2, zx^2 >$ and S is the surface of the region bounded by the paraboloid $y = x^2 + z^2$ and the planes $y = 1$ and $y = 4$ and $\boldsymbol{n}$ being the outward pointing normal vector.

$\text{div} F = 0 + z^2 + x^2 = z^2 + x^2$

$\iint_S \bar{F} \cdot \bar{n} ds = \iiint_V z^2 + x^2 \, dV$



$= \iint_D \int_1^4 x^2 + z^2 \, dy \, dA$

$= \iint_D (x^2+z^2)(3) dA$ let $x = r\cos\theta$, $z = r\sin\theta$ $1 \le r \le 2$, $0 \le \theta \le 2\pi$

$= 3\int_0^{2\pi}\int_1^2 r^2 \, r dr d\theta$

$= 3\int_0^{2\pi} \frac{r^4}{4}\Big|_1^2 d\theta = 3\left(\frac{2^4}{4} - \frac{1}{4}\right) 2\pi$

$= 3\left(4 - \frac{1}{4}\right) 2\pi$

$= 3\left(\frac{15}{4}\right) 2\pi = \frac{3\pi}{2}(15)$

Problem 1 continued.

$$= 2\int_0^{2\pi}\int_0^{\pi}\int_0^1 \rho^3 (\cos\theta\sin^2\phi + \sin\theta\sin^2\phi + \cos\phi\sin\phi)\,d\rho\,d\phi\,d\theta$$

$\downarrow \rho^4/4\big|_0^1$

$$= 1/2 \int_0^{2\pi}\int_0^{\pi} \sin^2\phi(\cos\theta + \sin\theta) + \cos\phi\sin\phi \; d\phi\,d\theta$$

$$= 1/2 \int_0^{2\pi}\int_0^{\pi} \tfrac{1}{2}(1-\cos 2\phi)(\cos\theta + \sin\theta) + \cos\phi\sin\phi \; d\phi\,d\theta$$

$$= 1/2 \int_0^{2\pi} \left(\tfrac{1}{2}\phi - \tfrac{1}{4}\sin 2\phi\right)(\cos\theta + \sin\theta) + \tfrac{1}{2}\sin^2\phi \;\Big|_0^{\pi} d\theta$$

$$= \frac{\pi}{4}\int_0^{2\pi} \cos\theta + \sin\theta \, d\theta = \frac{\pi}{4}(\sin\theta - \cos\theta\Big|_0^{2\pi} = 0 \checkmark$$