Math 13 Worksheet #15: Surface integrals of scalar functions

(1) True or false:

(a) The result of integrating a function over a surface is a scalar.

True

(b) For a region R in the xy–plane, $dS = dA$.

No

(2) Find the surface area of S, where S is the portion of the surface determined by $x = 9 - y^2 - z^2$ that lies on the positive side of the yz–plane (i.e., where $x \geq 0$).

$D = \{(y,z) : y^2 + z^2 \leq 9\}$.

let $\bar{r}(r,\theta) = \langle 9 - r^2, r\cos\theta, r\sin\theta \rangle$ $\quad 0 \leq \theta \leq 2\pi \quad 0 \leq r \leq 3$

$\bar{r}_r = \langle -2r, \cos\theta, \sin\theta \rangle \quad \bar{r}_\theta = \langle 0, -r\sin\theta, r\cos\theta \rangle$

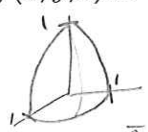
$$\bar{r}_r \times \bar{r}_\theta = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ -2r & \cos\theta & \sin\theta \\ 0 & -r\sin\theta & r\cos\theta \end{vmatrix} = \langle r\cos^2\theta + r\sin^2\theta, +2r^2\cos\theta, 2r^2\sin\theta \rangle$$

$u = 1 + 4r^2 \quad du = 8r\,dr$

$|\bar{r}_r \times \bar{r}_\theta| = \sqrt{r^2 + 4r^4} = r\sqrt{1+4r^2}$

$$S(A) = \int_0^{2\pi}\int_0^3 r\sqrt{1+4r^2}\,dr\,d\theta = \frac{2\pi}{8}\int_1^{37} u^{1/2}\,du = \frac{\pi}{4} u^{3/2}\left(\tfrac{2}{3}\right)\Big|_1^{37} = \frac{\pi}{6}\left(37^{3/2} - 1\right)$$

(3) Evaluate $\iint_S f(x,y,z)\,dS$ where $f(x,y,z) = e^z$ and S is the portion of unit sphere in the first octant.



easiest way to parameterize is in spherical.

$0 \leq \theta \leq \pi/2 \quad 0 \leq \phi \leq \pi/2$

$\bar{r}(\theta,\phi) = \langle \cos\theta\sin\phi, \sin\theta\sin\phi, \cos\phi \rangle$

$\bar{r}_\theta = \langle -\sin\theta\sin\phi, \cos\theta\sin\phi, 0 \rangle \quad \bar{r}_\phi = \langle \cos\theta\cos\phi, \sin\theta\cos\phi, -\sin\phi \rangle$

$$\bar{r}_\theta \times \bar{r}_\phi = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ -\sin\theta\sin\phi & \cos\theta\sin\phi & 0 \\ \cos\theta\cos\phi & \sin\theta\cos\phi & -\sin\phi \end{vmatrix} = \langle -\cos\theta\sin^2\phi, -(\sin\theta\sin^2\phi), -\sin^2\theta\sin\phi\cos\phi - \cos^2\theta\cos\phi\sin\phi \rangle$$

$|\bar{r}_\theta \times \bar{r}_\phi| = \sqrt{\cos^2\theta\sin^4\phi + \sin^2\theta\sin^4\phi + \cos^2\phi\sin^2\phi} = \sqrt{\sin^4\phi + \cos^2\phi\sin^2\phi} = \sin\phi$

$$\iint_S f\,dS = \int_0^{\pi/2}\int_0^{\pi/2} e^{\cos\phi}\sin\phi\,d\phi\,d\theta = \pi/2\left(-e^{\cos\phi}\Big|_0^{\pi/2}\right) = \frac{\pi}{2}\left(-e^0 + e\right) = \frac{\pi}{2}(e-1)$$

(4) Evaluate $\iint_S f(x,y,z)\,dS$ where $f(x,y,z) = x - z + y^2$ and S is given by $r(u,v) = \langle u+v, 2\sqrt{u^2+v^2}, u - v \rangle$ on the region in the uv–plane bounded by the graphs of $v = u$ and $v = u^2$.

See next page.

Step 1 Compute Jacobian

$\bar{r}(u,v) = \langle u+v,\ 2\sqrt{u^2+v^2},\ u-v \rangle$

$\bar{r}_u = \langle 1,\ \frac{2u}{\sqrt{u^2+v^2}},\ 1 \rangle$

$\bar{r}_v = \langle 1,\ \frac{2v}{\sqrt{u^2+v^2}},\ -1 \rangle$

$$\bar{r}_u \times \bar{r}_v = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ 1 & \frac{2u}{\sqrt{u^2+v^2}} & 1 \\ 1 & \frac{2v}{\sqrt{u^2+v^2}} & -1 \end{vmatrix}$$

$$= \left\langle \frac{-2(u+v)}{\sqrt{u^2+v^2}},\ -\cancel{(-1-1)}^{\,2},\ \frac{2(v-u)}{\sqrt{u^2+v^2}} \right\rangle$$

$$|\bar{r}_u \times \bar{r}_v| = \left[\frac{4(u+v)^2}{u^2+v^2} + 4 + 4\frac{(v-u)^2}{u^2+v^2}\right]^{1/2}$$

$$= 2\left(\frac{u^2+v^2}{u^2+v^2} + \frac{2uv}{u^2+v^2} + 1 + \frac{v^2+u^2}{u^2+v^2} - \frac{2uv}{u^2+v^2}\right)^{1/2}$$

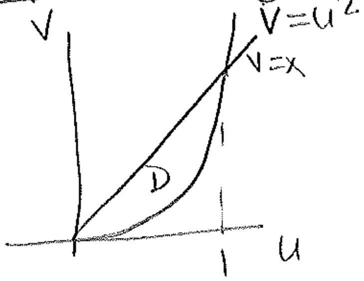
$$= 2\left(3 + \frac{2uv}{u^2+v^2} - \frac{2uv}{u^2+v^2}\right)^{1/2} = 2\sqrt{3}.$$

Step 2 Transform f

$f(x,y,z) = f(\bar{r}(u,v)) = u+v-u+v+4(u^2+v^2)$

$= 2v + 4u^2 + 4v^2$

Step3 Integrate.



$$\iint_S f\,dS = \iint_D f(\bar r(u,v))\,|\bar r_u \times \bar r_v|\,dA$$

$$= \int_0^1\int_{u^2}^{u} \left[2v+4u^2+4v^2\right] 2\sqrt{3}\,dv\,du.$$

$$= \int_0^1 v^2+4u^2v+\frac{4}{3}v^3 \Big|_{u^2}^{u} (2\sqrt{3})\,du$$

$$= 2\sqrt{3}\int_0^1 u^2+4u^3+\frac{4}{3}u^3-\left(u^4+4u^4+\frac{4}{3}u^6\right) du$$

$$= 2\sqrt{3}\int_0^1 u^2+\frac{16}{3}u^3-5u^4-\frac{4}{3}u^6\,du$$

$$= 2\sqrt{3}\left[\frac{u^3}{3}+\frac{4}{3}u^4-u^5-\frac{4}{21}u^7\Big|_0^1\right]$$

$$= 2\sqrt{3}\left(1/3+4/3-1-4/21\right)$$