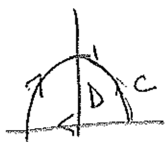
Math 13 Worksheet #13: Green's Theorem

(1) Use Green's Theorem to evaluate $\int_C \boldsymbol{F}\cdot d\boldsymbol{r}$, where $\boldsymbol{F} = < y, 2x >$ and C is the boundary of the region bounded by the x-axis and the curve $y = 1-x^2$, transversed in the clockwise direction.

Since Clockwise



$$\int_C \bar{F}\cdot d\bar{r} = -\iint_D (Q_x - P_y)\,dA = \int_{-1}^{1}\int_0^{1-x^2} (2-1)\,dy\,dx$$

$$= \int_{-1}^{1} 1-x^2\,dx = x - \frac{x^3}{3}\Big|_{-1}^{1} = 1 - 1/3 - (-1 + 1/3)$$

$$= 2 - 2/3 = 1\,1/3$$

(2) Evaluate $\int_C \boldsymbol{F}\cdot d\boldsymbol{r}$ where $\boldsymbol{F} = < \frac{1}{2}x^2y^3, xy >$ and C is the circle with radius 3, centered at the origin transversed clockwise.

$$\int_C \bar{F}\cdot d\bar{r} = -\iint_D (Q_x - P_y)\,dA = -\iint_D (y - 3/2\,x^2y^2)\,dA$$

$$= -\int_0^{2\pi}\int_0^3 (r\sin\theta - r^4 \tfrac{3}{2}\cos^2\theta\sin^2\theta)\,r\,dr\,d\theta$$

$$= -\int_0^{2\pi} \left(\frac{r^3}{3}\sin\theta - \frac{r^6}{6}\,\frac{3}{2}\cos^2\theta\sin^2\theta\right)\Big|_0^3 d\theta$$

$$= -\int_0^{2\pi} 9\sin\theta - \frac{3^6}{4}\underbrace{\cos^2\theta\sin^2\theta}_{\left(\frac{\sin 2\theta}{2}\right)^2}\,d\theta$$

continued on next page.

(3) Evaluate $\iint_R (3xy - 4x^2y)\,dA$ where R is the unit disk.

$Q_x \quad P_y$

$$\Rightarrow Q = \frac{3x^2y}{2} \qquad P = 2x^2y^2 \qquad \Rightarrow \bar{F} = \left< 2x^2y^2, \frac{3x^2y}{2} \right>$$

$$\iint_R (3xy - 4x^2y)\,dA = \int_C \bar{F}\cdot d\bar{r}$$

$$= \int_0^{2\pi} < 2\cos^2 t\sin^2 t, \tfrac{3}{2}\cos^2 t\sin t > \cdot < -\sin t, \cos t > dt$$

$$= \int_0^{2\pi} (-2\cos^2 t\sin^3 t + \tfrac{3}{2}\cos^3 t\sin t)\,dt$$

$$= -\frac{1}{120}\left(\cos^3 t\,(45\cos t + 24\cos 2t - 56)\right)\Big|_0^{2\pi} = 0$$

2 continued

$$-\int_0^{2\pi}\left(9\sin\theta - \frac{3^6}{16}(\sin(2\theta))^2\right)d\theta$$

$$= -\int_0^{2\pi}\left(9\sin\theta - \frac{3^6}{16}\frac{1}{2}(1-\cos(4\theta))\right)d\theta$$

$$= -\left(9\cos\theta - \frac{3^6}{16}\frac{1}{2}\left(\theta - \frac{1}{4}\sin(4\theta)\right)\right)\Big|_0^{2\pi}$$

$$= \frac{3^6}{16}\pi$$