Math 13, Spring 2014 – Homework Solutions Week 8

(1) (Chapter 16.7, Problem #4) Suppose that $f(x,y,z) = g(\sqrt{x^2+y^2+z^2})$, where g is a function of one variable such that $g(2) = -5$. Evaluate $\int_S f(x,y,z)\, dS$, where S is the sphere $x^2+y^2+z^2 = 4$.

Solution. $\int_S f(x,y,z)\, dS = \int_S g(\sqrt{4})\, dS = -5 \cdot \int_S dS = -5 \cdot 16\pi = -80\pi$.

(2) (Chapter 16.7, Problem #39) Find the center of mass of the hemisphere $x^2+y^2+z^2 = a^2$, $z \geq 0$, if it has constant density.

Solution. The surface is symmetric around the z-axis so its center of mass will have coordinates $(0,0,\bar{z})$. We can omit the density k from the integrals since it will disappear when we compute $\bar{z}$. The integral $\int_S z\, dS$ can be done by parametrizing the surface as a function, $z = \sqrt{a^2-x^2-y^2}$. Then

$$dS = \sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}\, dA = \sqrt{\frac{a^2}{a^2-x^2-y^2}}\, dA$$

so the integral is $\iint_D a\, dA = a \cdot \pi a^2 = \pi a^3$. The mass (with density $k=1$) of the hemisphere equals $2\pi a^2$, so $\bar{z} = \dfrac{\pi a^3}{2\pi a^2} = \dfrac{a}{2}$.

(3) (Chapter 16.7, Problem #29) Let $\mathbf{F} = \langle x, 2y, 3z \rangle$, and let S be the cube with vertices $(\pm 1, \pm 1, \pm 1)$ with positive orientation. Evaluate $\int_S \mathbf{F} \cdot d\mathbf{S}$.

Solution. Each face of the cube will have a normal vector $\pm\mathbf{i}$, $\pm\mathbf{j}$, or $\pm\mathbf{k}$, and each face has area 4. The flux integral equals the sum over all the faces of $\int_{face} \mathbf{F} \cdot \mathbf{n}\, dS = 1\cdot 4 + 1 \cdot 4 + 2 \cdot 4 + 2 \cdot 4 + 3 \cdot 4 + 3 \cdot 4 = 48$.

(4) (Chapter 16.7, Problem #49) Let F be an inverse square vector field (ie, $\mathbf{F}(\mathbf{r}) = \dfrac{c\mathbf{r}}{|\mathbf{r}|^3}$ for some constant c, where $\mathbf{r} = \langle x, y, z \rangle$. Show that the flux of F across a sphere S centered at the origin is independent of the radius of S.

Solution. The quantity $|\mathbf{r}|$ is the radius of the sphere, which we may write as the constant a. We can write $\mathbf{F} = \dfrac{c}{a^2}\left\langle \dfrac{x}{a}, \dfrac{y}{a}, \dfrac{z}{a} \right\rangle$. The vector part of this expression is a unit vector normal to the sphere.

$$\begin{aligned}
\iint_S \mathbf{F} \cdot \mathbf{n}\, dS &= \frac{c}{a^2} \left\langle \frac{x}{a}, \frac{y}{a}, \frac{z}{a} \right\rangle \cdot \left\langle \frac{x}{a}, \frac{y}{a}, \frac{z}{a} \right\rangle dS \\
&= \iint_S \frac{c}{a^2} \cdot \frac{a^2}{a^2}\, dS \\
&= \frac{c}{a^2} \cdot 4\pi a^2 \\
&= 4c\pi
\end{aligned}$$

The result is constant, not dependent on the radius a.