Homework #7

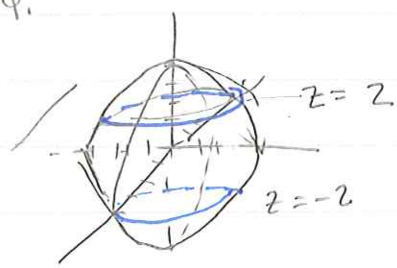
1) (Problem #24, §16.6)

sphere $x^2+y^2+z^2=16$ $\Rightarrow \rho^2=16 \Rightarrow \rho=4.$

$x = 4\cos\theta\sin\phi$

$y = 4\sin\theta\sin\phi$

$z = 4\cos\phi.$



$0 \le \theta \le 2\pi$ Now find bounds for ϕ.

lowerbound $z = 4\cos\phi = 2 \rightarrow \cos\phi = 1/2 \rightarrow \phi = \pi/3$

Upperbound $z = 4\cos\phi = -2 \rightarrow \cos\phi = -1/2 \rightarrow \phi = 2\pi/3$

$\pi/3 \le \phi \le 2\pi/3$.

So the parametric representation is

$x = 4\cos\theta\sin\phi$ $\quad 0 \le \theta \le 2\pi$

$y = 4\sin\theta\sin\phi$ $\quad \pi/3 \le \phi \le 2\pi/3$

$z = 4\cos\phi$

2) $\bar{r} = \langle \sin u,\ \cos u \sin v,\ \sin v \rangle$

Goal: Find tangent plane at $u = \pi/6$, $v = \pi/6$.

pt on surface $P = r(\pi/6, \pi/6) = (\sin \pi/6, \cos\pi/6 \sin\pi/6, \sin\pi/6)$
$= (1/2, \sqrt{3}/4, 1/2)$

To find normal vector we need Tangent vectors in the direction of u & v.

$\bar{r}_u = \langle \cos u, -\sin u \sin v, 0 \rangle$

$\bar{r}_u(\pi/6, \pi/6) = \langle \sqrt{3}/2, -1/4, 0 \rangle$

$\bar{r}_v = \langle 0, \cos u \cos v, \cos v \rangle$

$\bar{r}_v(\pi/6, \pi/6) = \langle 0, 3/4, \sqrt{3}/2 \rangle$

Then the normal vector $\bar{n}$ is given by

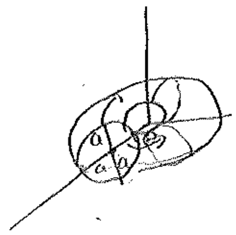
$$\bar{n} = r_u \times r_v = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \sqrt{3}/2 & -1/4 & 0 \\ 0 & 3/4 & \sqrt{3}/2 \end{vmatrix} =$$

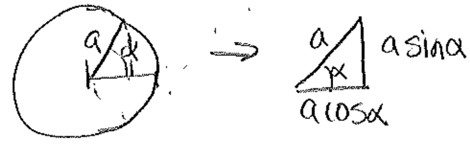
$$= \bar{i}\left(-\sqrt{3}/8 - 0\right) - \bar{j}\left(3/4 - 0\right) + \bar{k}\left(3\sqrt{3}/8\right)$$

$$= \langle -\sqrt{3}/8, -3/4, 3\sqrt{3}/8 \rangle$$

Eqn of Tangent plane: $\bar{n} \cdot \langle (x, y, z) - P \rangle = 0$

$$-\sqrt{3}/8\,(x - 1/2) - 3/4\,(y - \sqrt{3}/4) + 3\sqrt{3}/8\,(z - 1/2) = 0.$$





3) The center of the torus is a circle with radius b in the xy-plane. ie.

$C(\theta) = \langle b\cos\theta, b\sin\theta, 0\rangle$.

The z coordinate at any pt on surface of torus is given by $z = a\sin\alpha$

Thus the magnitude of (x,y) in the xy-plane must be $|a\cos\alpha|$ (see triangle above)

Since we know the x & y components revolve wrt θ as in C, the circle making up the surface at any θ is given by

$\langle a\cos\theta\cos\alpha, a\sin\theta\cos\alpha, a\sin\alpha\rangle$

Putting together the two curves we find the surface is given by

$$\bar{r}(\theta,\alpha) = \langle b\cos\theta + a\cos\theta\cos\alpha,\ b\sin\theta + a\sin\theta\cos\alpha,\ a\sin\alpha\rangle$$

where $0 \le \theta, \alpha \le 2\pi$.

4) Surface Area

$A(S) = \iint_D |\bar{r}_\theta \times \bar{r}_\alpha| \, dA.$

$\bar{r}_\theta = \langle -b\sin\theta - a\sin\theta\cos\alpha,\ b\cos\theta + a\cos\theta\cos\alpha,\ 0 \rangle$

$\bar{r}_\alpha = \langle -a\cos\theta\sin\alpha,\ -a\sin\theta\sin\alpha,\ a\cos\alpha \rangle$

$$\bar{r}_\theta \times \bar{r}_\alpha = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ -b\sin\theta - a\sin\theta\cos\alpha & b\cos\theta + a\cos\theta\cos\alpha & 0 \\ -a\cos\theta\sin\alpha & -a\sin\theta\sin\alpha & a\cos\alpha \end{vmatrix}$$

3^{rd} part $= \bar{k}\left[(-b\sin\theta - a\sin\theta\cos\alpha)(-a\sin\theta\sin\alpha) + a\cos\theta\sin\alpha\,(b\cos\theta + a\cos\theta\cos\alpha)\right]$

$= \bar{k}(+ab\sin^2\theta\sin\alpha + a^2\sin^2\theta\cos\alpha\sin\alpha + ab\cos^2\theta\sin\alpha + a^2\cos^2\theta\cos\alpha\sin\alpha.)$

$= \bar{k}\, a\sin\alpha(b + a\cos\alpha)$

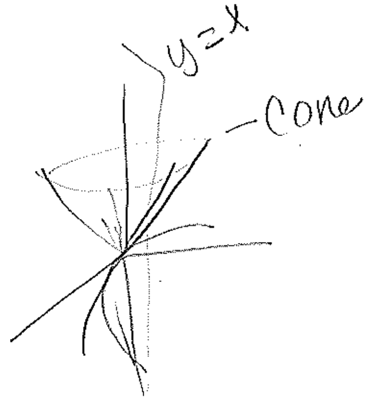
1^{st} part $\bar{i}(ab\cos\theta\cos\alpha + a^2\cos\theta\cos^2\alpha) + \bar{j}(-ab\sin\theta\cos\alpha - a^2\sin\theta\cos^2\alpha)$

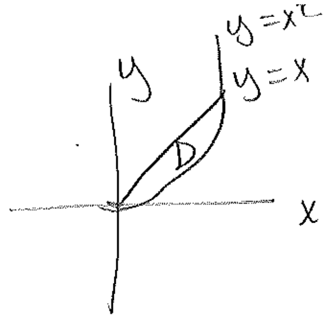
$\Rightarrow \bar{r}_\theta \times \bar{r}_\alpha = \langle a\cos\alpha\cos\theta(b + a\cos\alpha),\ a\sin\theta\cos\alpha(b + a\cos\alpha),\ a\sin\alpha(b + a\cos\alpha) \rangle$

$|\bar{r}_\theta \times \bar{r}_\alpha| = a(b + a\cos\alpha)$

$A(S) = \int_0^{2\pi}\int_0^{2\pi} a(b + a\cos\alpha)\, d\alpha\, d\theta = 4\pi^2 ab$

5)





We can parameterize the surface by

$$\bar{r}(u,v) = \langle u, v, \sqrt{u^2+v^2} \rangle$$

$$0 \le u \le 1 \quad u^2 \le v \le u$$

now we need $\bar{r}_u \times \bar{r}_v$

$$\bar{r}_u = \langle 1, 0, \tfrac{1}{2}(2u)(u^2+v^2)^{-1/2} \rangle$$

$$\bar{r}_v = \langle 0, 1, v(u^2+v^2)^{-1/2} \rangle$$

$$\bar{r}_u \times \bar{r}_v = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ 1 & 0 & u(u^2+v^2)^{-1/2} \\ 0 & 1 & v(u^2+v^2)^{-1/2} \end{vmatrix}$$

$$= \bar{i}\left(-u(u^2+v^2)^{-1/2}\right) - \bar{j}\, v(u^2+v^2)^{-1/2} + \bar{k}$$

$$|\bar{r}_u \times \bar{r}_v| = \left[u^2(u^2+v^2)^{-1} + v^2(u^2+v^2)^{-1} + 1\right]^{1/2} = \sqrt{2}$$

$$A(S) = \int_0^1 \int_{u^2}^{u} \sqrt{2}\, dv\, du = \sqrt{2} \int_0^1 u - u^2\, du$$

$$= \sqrt{2}\left[\frac{u^2}{2} - \frac{u^3}{3}\Big|_0^1\right] = \sqrt{2}\left(1/2 - 1/3\right) = \sqrt{2}/6$$