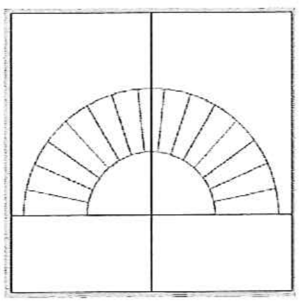
Math 13, Spring 2014 – Homework Solutions Week 2

(Problem #13, Chapter 15.5) The boundary of a lamina consists of the semicircles $y = \sqrt{1-x^2}$ and $y = \sqrt{4-x^2}$ together with the portion of the x-axis that joins them. Find the center of mass of the lamina if the density at any point is proportional to its distance from the origin.

Solution. Here is a sketch of the lamina.



In rectangular coordinates, the density is porportional to the distance of a point (x, y) to the origin: $\rho(x, y) = k\sqrt{x^2+y^2}$.

Both the lamina and the function being integrated become much simpler in polar coordinates, so it makes sense to set up the integrals this way.

$$\begin{aligned} m &= \int_0^\pi \int_1^2 kr\, r\, dr\, d\theta \qquad (1) \\ &= \int_0^\pi \int_1^2 kr^2\, dr\, d\theta \\ &= \int_0^\pi k\frac{r^3}{3}\Big|_1^2 \, d\theta \\ &= \int_0^\pi k\frac{7}{3}\, d\theta \\ &= k\frac{7}{3}\theta\Big|_0^\pi \\ &= k\frac{7}{3}\pi \end{aligned}$$

$$
\begin{aligned}
M_x &= \int_0^{\pi} \int_1^2 r\sin(\theta) \cdot kr\, r\, dr\, d\theta \\
&= \int_0^{\pi} \int_1^2 \sin(\theta) \cdot kr^3\, dr\, d\theta \\
&= \int_0^{\pi} \sin(\theta) \cdot k\frac{r^4}{4}\bigg|_1^2\, d\theta \\
&= \int_0^{\pi} \sin(\theta) \cdot k\frac{15}{4}\, d\theta \\
&= k\frac{15}{4} \cdot (-\cos(\theta))\bigg|_0^{\pi} \\
&= k\frac{15}{2}
\end{aligned} \tag{2}
$$

$$
\begin{aligned}
M_y &= \int_0^{\pi} \int_1^2 r\cos(\theta) \cdot kr\, r\, dr\, d\theta \\
&= \int_0^{\pi} \int_1^2 \cos(\theta) \cdot kr^3\, dr\, d\theta \\
&= \int_0^{\pi} \cos(\theta) \cdot k\frac{r^4}{4}\bigg|_1^2\, d\theta \\
&= \int_0^{\pi} \cos(\theta) \cdot k\frac{15}{4}\, d\theta \\
&= k\frac{15}{4} \cdot (\sin(\theta))\bigg|_0^{\pi} \\
&= 0
\end{aligned} \tag{3}
$$

The center of mass is

$$(\bar{x}, \bar{y}) = \left(\frac{M_y}{m}, \frac{M_x}{m}\right) = \left(0, \frac{45}{14\pi}\right)$$

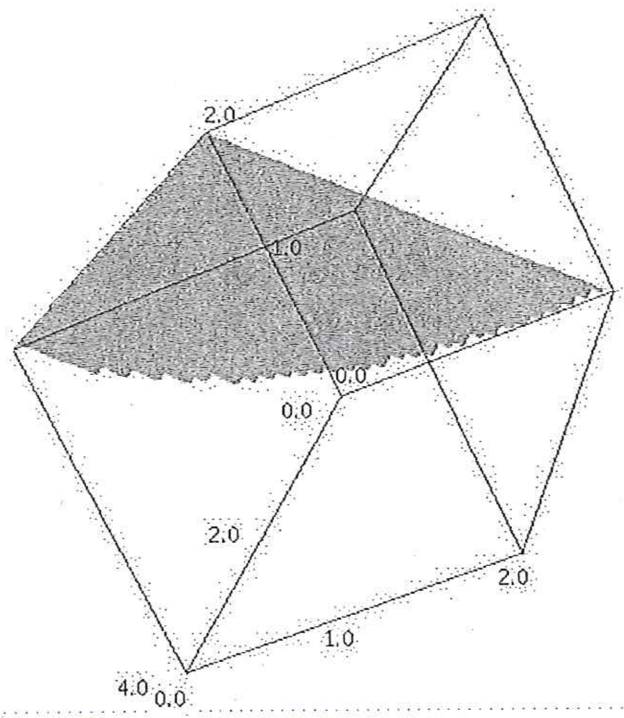
(Problem #28, Chapter 15.7) Sketch the solid whose volume is given by the following iterated integral, and compute the value of that volume:

$$\int_0^2 \int_0^{2-y} \int_0^{4-y^2} dx\, dz\, dy.$$

This integral is equal to

$$\int_0^2 \int_0^{2-y} 4-y^2\, dz\, dy = \int_0^2 (4-y^2)(2-y)\, dy = \int_0^2 8-4y-2y^2+y^3\, dy = 16-8-\frac{16}{3}+4 = \frac{20}{3}.$$

A poorly drawn picture of the top part of this surface is given by the plot below; it lies over the region $0 \le y \le 2, 0 \le x \le 4-y^2$ in the xy-plane.



(Problem #33, Chapter 15.7) (Consult the textbook for a useful figure.) The figure shows the region of integration for the integral

$$\int_0^1 \int_{\sqrt{x}}^1 \int_0^{1-y} f(x,y,z)\, dz\, dy\, dx.$$

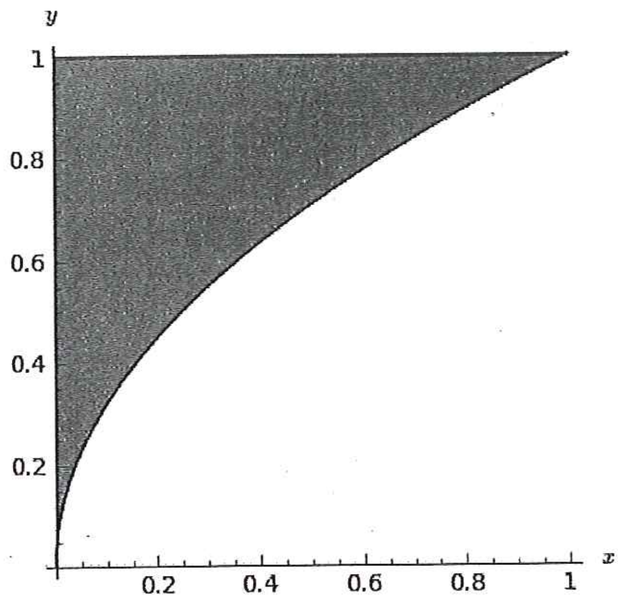
Rewrite this integral as an equivalent iterated integral in the five other orders. Give some reasons for why your answers are correct (in particular, give a brief explanation of how you calculate projections of this region to the various coordinate planes).

Solution. Let E be the region of integration. To solve this problem, we will want to compute the projection of E onto the xy, xz, and yz-planes. Notice that the given order of integration and the bounds on those integrals tells us that E is defined by inequalities

$$0 \leq z \leq 1-y, \sqrt{x} \leq y \leq 1, 0 \leq x \leq 1.$$

This already tells us that the projection of E onto the xy plane is described by inequalities $\sqrt{x} \leq y \leq 1, 0 \leq x \leq 1$. (See the figure below.) This region is also described by inequalities $0 \leq y \leq 1, 0 \leq x \leq y^2$. Therefore, for order of integration $dz\,dx\,dy$, we have bounds

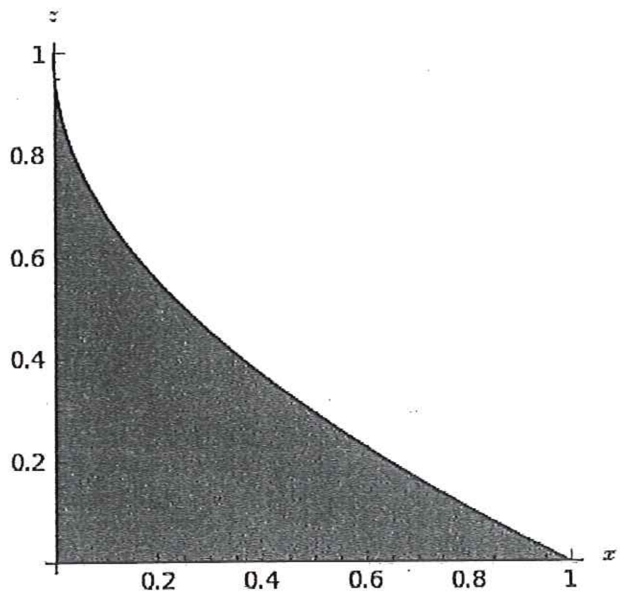
$$\int_0^1 \int_0^{x^2} \int_0^{1-y} f(x,y,z)\,dz\,dx\,dy.$$



Consider the projection onto the xz-plane. To determine this region, we want to know for what ordered pairs (x,z) does the set of inequalities $0 \leq z \leq 1-y, \sqrt{x} \leq y \leq 1, 0 \leq x \leq 1$ have a solution in y. (Indeed, it is exactly when this happens that we have a point (x,y,z) in E whose projection to the xz-plane is (x,z).) Out of this set of inequalities, the ones that involve y are $z \leq 1-y, \sqrt{x} \leq y, y \leq 1$. Notice that because we already know $0 \leq x, z \leq 1$ must already be true, it is the case that $z \leq 1-y$, which is the same as $y \leq 1-z$, implies $y \leq 1$. Therefore, we want to know for what ordered pairs (x,z) the inequalities $\sqrt{x} \leq y \leq 1-z$ has a solution. This evidently happens exactly when $\sqrt{x} \leq 1-z$. The graph of this region is shown below.

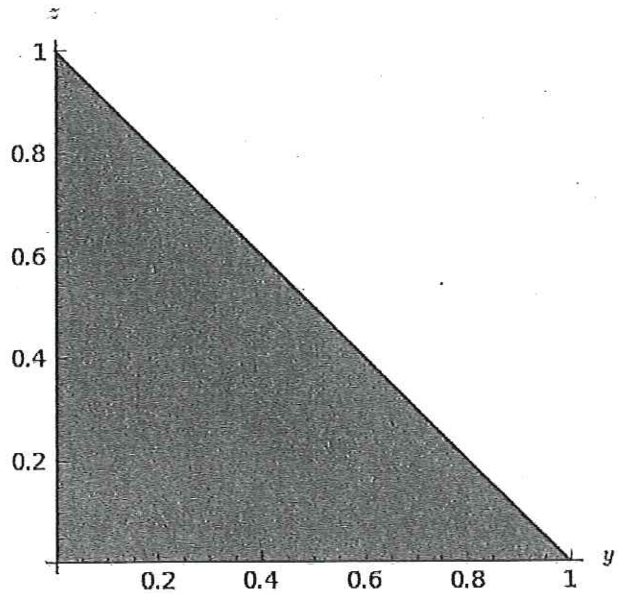
We can either express this region with inequalities $0 \leq x \leq (1-z)^2, 0 \leq z \leq 1$, or $0 \leq z \leq 1-\sqrt{x}, 0 \leq x \leq 1$. In addition, we already saw that $\sqrt{x} \leq y \leq 1-z$, so the original integral also equals

$$\int_0^1 \int_0^{(1-z)^2} \int_{\sqrt{x}}^{1-z} f(x,y,z)\, dy\, dx\, dz = \int_0^1 \int_0^{1-\sqrt{x}} \int_{\sqrt{x}}^{1-z} f(x,y,z)\, dy\, dz\, dx.$$



Finally, consider the projection onto the yz-plane. It is fairly clear from the drawing that this region is defined by $0 \le z \le 1-y, 0 \le y \le 1$, and algebraically we can see this because we know that $0 \le z \le 1-y$, and for any ordered pair (x,y), there will be some x satisfying $\sqrt{x} \le y, 0 \le x$. We also see that $0 \le x \le y^2$ are the bounds for x. Therefore, the original integral is equal to

$$\int_0^1 \int_0^{1-y} \int_0^{y^2} f(x,y,z)\, dx\, dz\, dy = \int_0^1 \int_0^{1-z} \int_0^{y^2} f(x,y,z)\, dx\, dy\, dz.$$



(4) (Problem #24, Chapter 15.8) Find the volume of the solid that lies between the paraboloid $z = x^2 + y^2$ and the sphere $x^2 + y^2 + z^2 = 2$.

Solution. Let E be the solid in question. A quick sketch of this region shows that the paraboloid $z = x^2 + y^2$ bounds this region from below and the sphere $x^2 + y^2 + z^2 = 2$ bounds this region from above. Furthermore, the projection of this region to the xy-plane will be a circle whose radius is equal to the radius of the circle where these two surfaces intersect (see picture below). To find this intersection, we can substitute $z = x^2 + y^2$ into $x^2 + y^2 + z^2 = 2$ to get $(x^2 + y^2) + (x^2 + y^2)^2 = 2$. One quickly checks this has solutions $x^2 + y^2 = -2, 1$. Since $x^2 + y^2 = -2$ is impossible, the circle of intersection is given by equations $x^2 + y^2 = z = 1$. This is a circle of radius 1, so the projection onto the xy-plane is the disc D given by $x^2 + y^2 \leq 1$.

Therefore, the triple integral which equals the volume of this solid is given by

$$\iiint_E dV = \iint_D \int_{x^2+y^2}^{\sqrt{2-x^2-y^2}} 1\,dz\,dA = \iint_D \sqrt{2-x^2-y^2} - (x^2+y^2)\,dA.$$

Use polar coordinates; this double integral is equal to

$$\int_0^{2\pi}\int_0^1 r(\sqrt{2-r^2} - r^2)\,dr = 2\pi\left(\frac{(2-r^2)^{3/2}\cdot -1}{3} - \frac{r^4}{4}\right)\Big|_{r=0}^{r=1} = \pi\left(\frac{8\sqrt{2}-7}{6}\right).$$

(A lot of routine intermediate steps were skipped above.) □

(Problem #28, Chapter 15.8) Find the mass of a ball B given by $x^2+y^2+z^2 \leq a^2$ if the density at any point is proportional to its distance from the z-axis. Your density function will have a constant in it; you can either keep the constant in your calculations or just assume the constant is 1. (Think about why this problem is better suited to cylindrical than spherical coordinates; you don't have to answer this in writing, but it is worth understanding why you use cylindrical coordinates here.)

Solution. The density function is given by $\rho(x,y,z) = \sqrt{x^2+y^2}$. Indeed, the distance of (x,y,z) to the z-axis is just the distance of (x,y,z) to the nearest point on the z-axis, which one can quickly check is $(0,0,z)$. Therefore we want to calculate

$$m = \iiint_B \sqrt{x^2+y^2}\,dV.$$

If we use cylindrical coordinates, then B is described by $0 \leq \theta \leq 2\pi, 0 \leq r \leq a, -\sqrt{a^2-r^2} \leq z \leq \sqrt{a^2-r^2}$. Therefore, we want to evaluate

$$\int_0^{2\pi}\int_0^a\int_{-\sqrt{a^2-r^2}}^{\sqrt{a^2-r^2}} r^2\,dz\,dr\,d\theta. = 2\pi\int_0^a r^2\sqrt{a^2-r^2}\cdot 2\,dr = 4\pi\int_0^a r^2\sqrt{a^2-r^2}\,dr.$$

At this point, you can either look up this integral in an integral table, or calculate it using trigonometric substitution. More precisely, make the substitution $r = a\sin\theta$. Then $dr = a\cos\theta d\theta$, so the integral above (with the 4π removed for now) is equal to

$$\int_{r=0}^{r=a} a^2\sin^2\theta\cdot a\cos\theta\cdot a\cos\theta\,d\theta = a^4\int_{r=0}^{r=a}\cos^2\theta\sin^2\theta\,d\theta.$$

To integrate an expression of this type, we use the identities $\cos^2\theta = \dfrac{1+\cos 2\theta}{2}, \sin^2\theta = \dfrac{1-\cos 2\theta}{2}$. We also convert the bounds $r=0, r=a$ to the corresponding bounds $\theta = 0, \theta = \pi/2$:

$$\frac{a^4}{4}\int_0^{\pi/2} 1-\cos^2 2\theta\, d\theta = \frac{a^4}{4}\int_0^{\pi/2} 1-\frac{1+\cos 4\theta}{2}\, d\theta = \frac{a^4}{4}\left(\frac{\theta}{2}-\frac{\cos 4\theta}{2}\right)\Big|_0^{\pi/2} = \frac{a^4\pi}{16}.$$

Multiplying back in that factor of 4π, we see that the final answer is $a^4\pi^2/4$. □

(Problem #28, Chapter 15.9) Find the average distance from a point in a ball of radius a to its center.

Solution. This problem is similar to a problem from the last homework. It is clear that we may assume the ball in the problem is centered at the origin; let B be this ball. The distance of a point (x, y, z) from the origin is $d(x, y, z) = \sqrt{x^2+y^2+z^2} = \rho$. Therefore, the average distance of points in this ball from the center is given by

$$\frac{\iiint\limits_B d(x,y,z)\, dV}{\iiint\limits_B 1\, dV}.$$

The denominator is the volume of the ball and hence is $4\pi a^3/3$. The numerator is equal to

$$\int_0^{\pi}\int_0^{2\pi}\int_0^{a} \rho\cdot\rho^2 \sin\phi\, d\rho\, d\theta\, d\phi = \left(\frac{\rho^4}{4}\Big|_{\rho=0}^{\rho=a}\right)\cdot 2\pi\cdot\left(-\cos\phi\Big|_{\phi=0}^{\phi=\pi}\right) = \frac{\rho^4}{4}\cdot 2\pi\cdot 2 = \pi a^4.$$

Therefore the average distance is $\pi a^4/(4\pi a^3/3) = 3a/4$. □