MATH 13 MIDTERM 2 STUDY GUIDE, SPRING 2011

This is meant to be a quick reference guide for the topics you might want to know for the second midterm. You should also know the material from the first midterm, because the content of this class is cumulative. This guide probably isn't comprehensive, but should cover most of what we studied in this class. There are no examples, so be sure to consult your old homeworks, notes, or textbook for those. Also, I have tried to make this guide accurate, but I cannot rule out the possibility of minor typos. If you think something is incorrect, feel free to ask.

1. Triple integration

- Integration over rectangular prisms
- Integration over more general three-dimensional solids
- How to interchange order of integration (Fubini's Theorem)
- Cylindrical coordinates
- Spherical coordinates
- Jacobian, change of variables formula (this covers both double and triple integrals)
- Applications of triple integration to finding volume, mass of solid, center of mass, etc.

2. Line integration, vector fields

- Parameterizing common curves (line segments, graphs of functions $y = f(x)$, circles, ellipses, etc)
- Calculating line integrals of scalar functions over curves
- Calculating line integrals of vector fields over curves
- The Fundamental Theorem of Calculus for line integrals
- The various properties of conservative vector fields
- How to apply the differential criterion $P_y = Q_x$ for being a conservative vector field, simply connected domains
- Applications of line integrals to calculating arc length, center of mass of a wire, work, etc.
- Green's Theorem
- How to calculate the divergence and curl of vector fields, and basic properties of div and curl.

3. Formulas

A non-comprehensive collection of formulas from this class:

- Polar coordinates: $x = r\cos\theta, y = r\sin\theta$, if D is described by $\alpha \le \theta \le \beta, g_1(\theta) \le r \le g_2(\theta)$, then

$$\iint\limits_D f(x,y)\,dA = \int_\alpha^\beta \int_{g_1(\theta)}^{g_2(\theta)} f(r\cos\theta, r\sin\theta) r\,dr\,d\theta.$$

- Cylindrical coordinates: $x = r\cos\theta, y = r\sin\theta, z = z$, if E is described by $\alpha \le \theta \le \beta, r_1 \le r \le r_2, z_1 \le z \le z_2$ (z_i may be a function of r, θ, while r_i might be a function of θ), then

$$\iiint\limits_E f(x,y,z)\,dV = \int_\alpha^\beta \int_{r_1}^{r_2} \int_{z_1}^{z_2} f(r\cos\theta, r\sin\theta, z) r\,dz\,dr\,d\theta.$$

- Spherical coordinates: $x = \rho\sin\phi\cos\theta, y = \rho\sin\phi\sin\theta, z = \rho\cos\phi$, if E is described by $\phi_1 \le \phi \le \phi_2, \ldots$, then

$$\iiint\limits_E f(x,y,z)\,dV = \int_{\phi_1}^{\phi_2} \int_{\theta_1}^{\theta_2} \int_{\rho_1}^{\rho_2} f(\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi)\rho^2 \sin\phi\,d\rho\,d\theta\,d\phi.$$

- Jacobian: If T is a function from the uv plane to the xy plane, with $x = x(u,v), y = y(u,v)$, then the Jacobian of T is defined to be
$$J(x,y) = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix} = x_u y_v - x_v y_u.$$
The absolute value of the Jacobian represents a local area magnification factor. There is a corresponding formula for a transformation from uvw space to xyz space, involving a 3×3 determinant.
- Change of variables formula: if S is some region in the uv plane, and $R = T(S)$ (for example, if T is the polar change of coordinates, and S is the rectangle $0 \le r \le 1, 0 \le \theta \le 2\pi$, then $T(S)$ is the unit disc centered at the origin), then
$$\iint\limits_{T(S)} f(x,y)\,dA = \iint\limits_S f(x(u,v), y(u,v))|J(x,y)|\,du\,dv.$$
There is a corresponding formula for change of variables from uvw space to xyz space.
- Line integrals of scalar functions: If C is a curve parameterized by $\mathbf{r}(t) = \langle x(t), y(t)\rangle, a \le t \le b$, and $f(x,y)$ some function on C, then
$$\int_C f(x,y)\,ds = \int_a^b f(\mathbf{r}(t))|\mathbf{r}'(t)|\,dt = \int_a^b f(x(t),y(t))\sqrt{x'(t)^2 + y'(t)^2}\,dt.$$
When $f(x,y) = 1$, this specializes to the formula for arc length of C. There is a corresponding formula for curves which lie in $\mathbb{R}^n$.
- Line integrals of vector fields: If C is a curve parameterized as above, and $\mathbf{F}(x,y)$ some vector field on C, then
$$\int_C \mathbf{F}\cdot d\mathbf{r} = \int_a^b \mathbf{F}(\mathbf{r}(t))\cdot\mathbf{r}'(t)\,dt = \int_a^b \mathbf{F}(x(t),y(t))\cdot\langle x'(t), y'(t)\rangle\,dt.$$
If $\mathbf{F}$ represents a force, this line integral equals the work this force does on a particle which moves along the path C.
- The Fundamental Theorem of Calculus for Line Integrals: if $\mathbf{F} = \nabla f$ on a curve C parameterized as above, then
$$\int_C \mathbf{F}\cdot d\mathbf{r} = f(\mathbf{r}(b)) - f(\mathbf{r}(a)).$$
- Conservative vector fields: their properties. A vector field $\mathbf{F}$ is defined to be conservative on a domain D if there exists a function f on D such that $\mathbf{F} = \nabla f$. This is equivalent to the following properties:

  - For any closed curve C in D, $\int_C \mathbf{F} \cdot d\mathbf{r} = 0$.
  - For any two curves C_1, C_2 in D which start and end at the same point, $\int_{C_1} \mathbf{F} \cdot d\mathbf{r} = \int_{C_2} \mathbf{F} \cdot d\mathbf{r}$.

  If a vector field $\mathbf{F} = \langle P, Q \rangle$ is conservative, then $P_y = Q_x$ in all of D. However, in general the converse is true only if D is simply connected. For $\mathbf{F}$ in $\mathbb{R}^3$ (or more generally, any simply connected region D) the condition $P_y = Q_x$ is replaced by $\nabla \times \mathbf{F} = \mathbf{0}$.
- Green's Theorem: If C is a simple closed curve in the plane, D its interior, $\mathbf{F} = \langle P, Q \rangle$ some C^1 vector field on D, and C is given the positive (counterclockwise) orientation, then

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \iint_D Q_x - P_y \, dA.$$

4. Strategies for solving problems

A list of strategies we have learned for solving various types of problems.

- If you run into an iterated integral which you can't seem to evaluate, try switching the order of integration or changing coordinate systems.
- Polar coordinates are probably useful when dealing with circles, sectors of circles, annuli, or other geometric figures related to circles.
- Cylindrical coordinates are good when dealing with cylinders, paraboloids, or cones.
- Spherical coordinates are good when dealing with spheres or pieces of spheres.
- When calculating the line integral of a vector field, you can sometimes apply the Fundamental Theorem of Calculus for line integrals (if you can find a potential function for $\mathbf{F}$, which is not always possible or feasible) to skip parameterization of the curve C.
- You can show that $\mathbf{F}$ is conservative on D in a variety of ways. You can either explicitly construct a potential function via 'partial integration', or check that $P_y = Q_x$ if D is simply connected.
- You can show that $\mathbf{F}$ is not conservative on D in a variety of ways. You can either check that $P_y \neq Q_x$, or find a closed curve in C for which the integral of $\mathbf{F}$ along C is not equal to 0, or show that no potential function exists by trying to do partial integration and arriving at a contradiction.
- When calculating the line integral of a vector field over a simple closed curve C, sometimes Green's Theorem can simplify the calculation. This is especially true if C is polygonal, like a rectangle, or some other curve which has somewhat complicated parameterization.